Energy-Stable Galerkin Reduced Order Models for Nonlinear Compressible Flow

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Motivation

- Despite improved algorithms and powerful supercomputers, "high-fidelity" models are often too expensive for use in a design or analysis setting.

- **Targeted application area in which this situation arises:** compressible cavity flow problem.

  → **Large Eddy Simulations (LES)** with very fine meshes and long times are required to predict accurately dynamic pressure loads in cavity.

These simulations take **weeks** even when run in parallel on state-of-the-art supercomputers!
Proper Orthogonal Decomposition (POD)/Galerkin Method to Model Reduction

**High-Fidelity Simulations:**
- Snapshot 1
- Snapshot 2
- ...
- Snapshot $K$

**Step 1:**
**Modal Decomposition (POD):**

\[ x(t) \approx \Phi_M x_M(t) \]

**Step 2:**
**Galerkin Projection of LTI FOM:**

\[ \Phi_M^T [\dot{x}(t) = Ax(t) + Bu(t)] \]

- **Snapshot matrix:** $X = (x^1, \ldots, x^K) \in \mathbb{R}^{N \times K}$
- **SVD:** $X = U \Sigma V^T$
- **Truncation:** $\Phi_M = (\phi_1, \ldots, \phi_M) = U(:, 1: M)$

**“Small” ROM LTI System:**

\[
\begin{align*}
\dot{x}_M(t) &= \Phi_M^T A \Phi_M x_M(t) + \Phi_M^T B u(t) \\
y_M(t) &= C \Phi_M x_M(t)
\end{align*}
\]

$N$ = # of dofs in high-fidelity simulation
$K$ = # of snapshots
$M$ = # of dofs in ROM ($M \ll N, M \ll K$)
Discrete vs. Continuous Galerkin Projection

**Discrete Projection**

- Governing PDEs
  \[ \dot{q} = \mathcal{L}q \]
- CFD model
  \[ \dot{q}_N = A_N q_N \]
- Discrete modal basis \( \Phi \)
- Projection of CFD model (matrix operation)
  \[ \dot{a}_M = \Phi^T A_N \Phi a_M \]
- ROM
  \[ \dot{a}_M = \Phi^T A_N \Phi a_M \]

**Continuous Projection**

- Governing PDEs
  \[ \dot{q} = \mathcal{L}q \]
- CFD model
  \[ \dot{q}_N = A_N q_N \]
- Continuous modal basis* \( \phi_j(x) \)
- Projection of governing PDEs (numerical integration)
  \[ \dot{a}_j = (\phi_j, \mathcal{L}\phi_k) a_k \]
- ROM
  \[ \dot{a}_j = (\phi_j, \mathcal{L}\phi_k) a_k \]

* Continuous functions space is defined using finite elements.

This talk focuses on

If PDEs are linear or have polynomial non-linearities, projection can be calculated in **offline stage** of MOR.
Stability Issues of POD/Galerkin ROMs

**Full Order Model (FOM)**
\[ \dot{q}(t) = \mathcal{L}q(t) + \mathcal{N}(q(t)) \]

**Reduced Order Model (ROM)**
\[ \dot{q}_M(t) = A_M q_M(t) + N_M(q_M(t)) \]

**Problem:** FOM stable \(\not\Rightarrow\) ROM stable!

- There is no *a priori* stability guarantee for POD/Galerkin ROMs.
- Stability of a ROM is commonly evaluated *a posteriori* – RISKY!
- Instability of POD/Galerkin ROMs is a real problem in some applications...

...e.g., compressible flows, high-Reynolds number flows.

**Top right:** FOM
**Bottom right:** ROM
Energy-Stability

• **Practical Definition:** Numerical solution does not “blow up” in finite time.

• **More Precise Definition:** Numerical discretization does not introduce any spurious instabilities inconsistent with natural instability modes supported by the governing continuous PDEs.

**Numerical solutions must maintain proper energy balance.**

• Stability of ROM is intimately tied to choice of **inner product** for the Galerkin projection.

• Stability-preserving inner product derived using the **energy method**:
  • Bounds numerical solution energy in a physical way.
  • Borrowed from spectral methods community.
  • Analysis is straightforward for ROMs constructed via **continuous projection**.

**Practical implication of energy-stability analysis:**
energy inner product ensures that any “bad” modes will not introduce spurious non-physical numerical instabilities into the Galerkin approximation.
Linearized Compressible Flow Equations

Energy-Stability for Linearized PDEs:
FOM linearly stable $\Rightarrow$ ROM built in energy inner product linearly stable ($Re(\lambda) < 0$)
(WCCM X talk and paper: Kalashnikova & Arunajatesan, 2012).

Linearized compressible Euler/Navier-Stokes equations are appropriate when a compressible fluid system can be described by small-amplitude perturbations about a steady-state mean flow.

- Linearization of full compressible Euler/Navier-Stokes equations obtained as follows:
  1. Decompose fluid field as steady mean plus unsteady fluctuation
     \[ q(x, t) = \bar{q}(x) + q'(x, t) \]
  2. Linearize full nonlinear compressible Navier-Stokes equations around steady mean to yield linear hyperbolic/incompletely parabolic system
     \[ \dot{q}' + A_i(\bar{q}) \frac{\partial q'}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ K_{ij}(\bar{q}) \frac{\partial q'}{\partial x_i} \right] = 0 \]
Linearized compressible Euler/Navier-Stokes equations are symmetrizable (Barone & Kalashnikova, 2009; Kalashnikova & Arunajatesan, 2012).

- There exists a symmetric positive definite matrix $H \equiv H(\bar{q})$ (system “symmetrizer”) s.t.:
  - The convective flux matrices $HA_i$ are symmetric
  - The following augmented viscosity matrix is symmetric positive semi-definite
    \[
    K^s = \begin{pmatrix}
    HK_{11} & HK_{12} & HK_{13} \\
    HK_{21} & HK_{22} & HK_{23} \\
    HK_{31} & HK_{32} & HK_{33}
    \end{pmatrix}
    \]

- **Symmetry Inner Product** (weighted $L^2$ inner product):
  \[
  (q_1, q_2)_H = \int_\Omega q_1 H q_2 d\Omega
  \]

- If ROM is built in **symmetry inner product**, Galerkin approximation will satisfy the same energy expression as continuous PDEs:
  \[
  ||q'_M(x, t)||_H \leq e^{\beta t} ||q'_M(x, 0)||_H \quad (\Rightarrow \frac{dE_M}{dt} \leq 0 \text{ for uniform base flow})
  \]
Symmetrizers for Several Hyperbolic/Incompletely Parabolic Systems

- **Wave equation**: \( \ddot{u} = a^2 \frac{\partial^2 u}{\partial x^2} \) or \( \dot{q} = A \frac{\partial q}{\partial x} \) where \( q = (\dot{u}, \frac{\partial u}{\partial x}) \) \( \Rightarrow H = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix} \)

- **Linearized shallow water equations**: \( \dot{q}' + A_i(q) \frac{\partial q'}{\partial x_i} = 0 \) \( \Rightarrow H = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

- **Linearized compressible Euler**: \( \dot{q}' + A_i(q) \frac{\partial q'}{\partial x_i} = 0 \) \( \Rightarrow H = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 \\ 0 & \alpha^2 \gamma \bar{\rho}^2 \bar{p} & \bar{\rho} \alpha^2 & 0 \\ 0 & 0 & (1+\alpha^2) \gamma \bar{p} \end{pmatrix} \)

- **Linearized compressible Navier-Stokes**: \( \dot{q}' + A_i(q) \frac{\partial q'}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ K_{ij}(q) \frac{\partial q'}{\partial x_i} \right] = 0 \)

\( \Rightarrow H = \begin{pmatrix} \bar{\rho} & 0 & 0 \\ 0 & \bar{\rho} R & 0 \\ 0 & \frac{RT}{\bar{\rho}} (\gamma - 1) & 0 \\ 0 & 0 & \frac{RT}{\bar{\rho}} \end{pmatrix} \)

Continuous Projection Implementation: “Spirit” Code

“Spirit” ROM Code = 3D parallel C++ POD/Galerkin test-bed ROM code that uses data-structures and eigensolvers from Trilinos to build energy-stable ROMs for compressible flow problems → stand-alone code that can be synchronized with any high-fidelity code!

- POD modes defined using piecewise smooth finite elements.
- Gauss quadrature rules of sufficient accuracy are used to compute exactly inner products with the help of the libmesh library.
- Physics in Spirit:
  - **Linearized compressible Euler** ($L^2$, energy inner product).
  - **Linearized compressible Navier-Stokes** ($L^2$, energy inner product).
  - **Nonlinear isentropic compressible Navier-Stokes** ($L^2$, stagnation energy, stagnation enthalpy inner product).
  - **Nonlinear compressible Navier-Stokes** ($L^2$, energy inner product).

“SIGMA CFD” High-Fidelity Code = Sandia in-house finite volume flow solver derived from LESLIE3D (Genin & Menon, 2010), an LES flow solver originally developed in the Computational Combustion Laboratory at Georgia Tech.

First, testing of ROMs for these physics
Numerical Experiment: 2D Inviscid Pressure Pulse

- Inviscid pulse in a uniform base flow (linear dynamics).
- High-fidelity simulation run on mesh with 3362 nodes, up to time $t = 0.01$ seconds.
- 200 snapshots of solution used to construct $M = 20$ mode ROM in $L^2$ and symmetry inner products.

$x_{M,i}(t)$ vs. $(q'_{CFD}, \phi_i)$ for $i = 1, 2$
Numerical Experiment: 2D Inviscid Pressure Pulse (cont’d)

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$p'$: High-fidelity

$p'$: Symmetry ROM

$p'$: $L^2$ ROM

Time of snapshot 160
Nonlinear Compressible Flow Equations

Energy-Stability for Nonlinear PDEs:
ROM built in energy inner product will preserve stability of an equilibrium point at 0 for the governing nonlinear system of PDEs (Rowley, 2004; Kalashnikova et al., 2014).

- Compressible isentropic Navier-Stokes equations (cold flows, moderate Mach #):

\[
\frac{Dh}{Dt} + (\gamma - 1)h \nabla \cdot \mathbf{u} = 0
\]
\[
\frac{Du}{Dt} + \nabla h - \frac{1}{Re} \Delta \mathbf{u} = 0
\]

- Full compressible Navier-Stokes equations:

\[
\rho \frac{Du}{Dt} + \frac{1}{\gamma M^2} \nabla (\rho T) - \frac{1}{Re} \nabla \cdot \mathbf{\tau} = 0
\]
\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0
\]
\[
\rho \frac{DT}{Dt} + (\gamma - 1)\rho T \nabla \cdot \mathbf{u} - \frac{\gamma}{Pr Re} \nabla \cdot (\kappa \nabla T) - \left(\frac{\gamma (\gamma - 1) M^2}{Re}\right) \nabla \mathbf{u} \cdot \mathbf{\tau} = 0
\]

\( h \) = enthalpy
\( \mathbf{u} \) = velocity vector
\( \rho \) = density
\( T \) = temperature
\( \mathbf{\tau} \) = viscous stress tensor
In (Rowley, 2004), Rowley et al. showed that energy inner product for the compressible isentropic Navier-Stokes equations can be defined following a transformation of these equations.

- Transformed compressible isentropic Navier-Stokes equations:
  
  \[
  \frac{Dc}{Dt} + \frac{\gamma - 1}{2} c \nabla \cdot \mathbf{u} = 0
  \]
  
  \[
  \frac{D\mathbf{u}}{Dt} + \frac{2}{\gamma - 1} c \nabla c - \frac{1}{\text{Re}} \Delta \mathbf{u} = 0
  \]

- Family of inner products:
  
  \[
  (q_1, q_2)_\alpha = \int_{\Omega} \left( \mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{2\alpha}{\gamma - 1} c_1 c_2 \right) d\Omega
  \]

  \[
  \alpha = \begin{cases} 
  1 & \Rightarrow ||q||_\alpha = \text{stagnation enthalpy} \\
  \frac{1}{\gamma} & \Rightarrow ||q||_\alpha = \text{stagnation energy}
  \end{cases}
  \]

\[c = \text{speed of sound} \quad (c^2 = (\gamma - 1)h)\]

\[\mathbf{u} = \text{velocity}\]

If Galerkin projection step of model reduction is performed in \(\alpha\) inner product, then the Galerkin projection will preserve the stability of an equilibrium point at the origin (Rowley, 2004).
Energy-Stable ROMs for Nonlinear Compressible Flow (Full NS)

Present work extends ideas in (Rowley, 2004) to **full compressible Navier-Stokes equations.**

**Requirement:** transformation/inner product yields PDEs with only polynomial non-linearities.

- First, full compressible Navier-Stokes equations are **transformed** into the following variables:

  \[ a = \sqrt{\rho}, \quad b = au, \quad d = ae \]

- Next, the following “**total energy**” inner product is defined:

  \[ (q_1, q_2)_{TE} = \int_{\Omega} (b_1 \cdot b_2 + a_1 d_2 + a_2 d_1) \, d\Omega \]

  → Norm induced by total energy inner product is the total energy of the fluid system:

  \[ ||q||_{TE} = \int_{\Omega} \left( \rho e + \frac{1}{2} \rho u_i u_i \right) \, d\Omega \]

If Galerkin projection step of model reduction is performed in total energy inner product, then the Galerkin projection will preserve the stability of an equilibrium point at the origin (Kalashnikova *et al.* , 2014)

契Transformed equations have only polynomial non-linearities (projection of which can be computed in offline stage of MOR and stored).

契Transformation introduces higher order polynomial non-linearities.

契Efficiency of online stage of MOR can be recovered using interpolation (e.g., DEIM, gappy POD).
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Numerical Experiment: Viscous Laminar Cavity

- Viscous cavity problem at $M = 0.6$, $Re = 1500$ (laminar regime).

- **High-fidelity simulation**: DNS based on full nonlinear compressible Navier-Stokes equations with 99,408 nodes (right).

- 500 snapshots collected, every $\Delta t_{snap} = 1 \times 10^{-4}$ seconds.

- Snapshots used to construct $M = 5$ mode ROM for nonlinear compressible Navier-Stokes equations in $L^2$ and total energy inner products.

- $M = 5$ mode POD bases capture $\approx 99\%$ of snapshot energy.

*Figure above*: viscous laminar cavity problem domain/mesh.
Numerical Experiment: Viscous Laminar Cavity (cont’d)

- $L^2$ ROM exhibited instability for $M > 5$ modes.

- In contrast, total energy ROM remained stable and agreed well with high-fidelity solution!

**Figure above:** $u$-component of velocity as a function of time $t$
Summary & Future Work

• A Galerkin model reduction approach in which the continuous equations are projected onto the basis modes in a continuous inner product is proposed.

• It is shown that the choice of inner product for the Galerkin projection step is crucial to stability of the ROM.
  • For linearized compressible flow, Galerkin projection in the “symmetry” inner product leads to a ROM that is energy-stable for any choice of basis.
  • For nonlinear compressible flow, an inner product that induces the total energy of the fluid system is developed. A ROM constructed in this inner product will preserve the stability of an equilibrium point at 0 for the system.

• Results are promising for a nonlinear problem involving compressible viscous laminar flow over an open cavity: a total energy ROM remains stable whereas an $L^2$ ROM exhibits an instability.

Ongoing/Future Work

• Improve efficiency of nonlinear total energy ROMs through interpolation (e.g., DEIM, gappy POD)

• Studies of predictive capabilities of ROMs (robustness w.r.t. parameter changes).
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Thank You! Questions?
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Some references on these ideas:


- I. Kalashnikova, S. Arunajatesan. A Stable Galerkin Reduced Order Model (ROM) for Compressible Flow, WCCM-2012-18407, 10th World Congress on Computational Mechanics (WCCM X), Sao Paulo, Brazil (2012).

References


