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From: Irina Kalashnikova[1,2]

Subject: Pressure Fluctuation Power Spectral Density (PSD) Model for Turbulent Boundary Layer

[1] Aerosciences Department, Sandia National Laboratories, Albuquerque, NM.
[2] Institute for Computational & Mathematical Engineering (iCME), Stanford University, Stanford, CA.
E-mail addresses: ikalash@sandia.gov, irinak@stanford.edu (I. Kalashnikova)

1 Introduction

The present document describes an approach for deriving a model of the power spectral density (PSD) for the fluctuating pressure in a turbulent flow. Ultimately, a model for compressible, transitional and fully turbulent pressure fluctuation loading to calculate a random vibration pressure field on an RB/RV during reentry conditions is sought. Both a power spectral density (PSD) as well as a probability density function (PDF) of the fluctuating pressure are required. The model summarized herein is a model for the fluctuating pressure PSD only (not a fluctuating pressure PDF), and would be an alternative (or perhaps a supplement) to the earlier "eigenvalue problem" model [10] and the "legacy" model [13].

The flow to be modeled (turbulent flow and laminar-to-turbulent transition in re-entry conditions) is extremely complex. One approach in building a model for this problem is to select a much simpler scenario, one for which data are available, as a starting point, and ultimately extend this simpler model. A canonical problem in this context is that of a turbulent boundary layer over a flat plate. Much wind tunnel data are available for the turbulent boundary layer [11]. A Reynolds Averaged Navier-Stokes (RANS) code is also available for computing the mean flow.

The approach in deriving the PSD is based primarily on the work of Lee *et. al* [1, 2], as well as earlier works [4, 5]. The derivations take as the starting point the incompressible Navier-Stokes equations, so adjustments may be needed in higher Mach number regimes ($Ma > 5$). Physical assumptions are applied to the governing equations describing this flow scenario (turbulent incompressible boundary layer), and mathematical expressions are extracted from these equations. The result is an analytical expression for the fluctuating pressure PSD, written in terms of integrals that can be evaluated with ease using numerical quadrature, given closure models for unclosed terms that appear in these expressions. The idea would be to close these terms using data obtained from the literature, and/or wind tunnel experimental data.

2 Pressure Power Spectral Density (PSD) for Turbulent Boundary Layer

The power spectral density (PSD) of the pressure is obtained from the fluctuating Poisson equation, following the approach of [1, 2, 4, 5]. Assume the flow is an incompressible three-dimensional (3D) flow. Then, the equation for the fluctuating pressure arises by taking the divergence of the momentum equation, using continuity to cancel terms, performing a Reynolds decomposition of each unknown into a mean and fluctuating term, and then subtracting the time-averaged equation. The result is the following Poisson equation:

$$\Delta p' = -2\overline{\rho}\frac{\partial \overline{U}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j}[\overline{\rho}(u'_i u'_j - \overline{u'_i u'_j})], \tag{1}$$

where summation over repeated indices is implied and $\Delta = \nabla^2$ is the standard Laplace operator. Lower case quantities with $'$ denote fluctuations; upper case quantities with bars denote mean quantities. Overlines (bars) in general denote time-averaging.

The source terms on the right hand side of (1) represent the mean-shear-turbulence (MT) interaction (also denoted LST for "linear source term"):

$$\mathrm{MT} \equiv \mathrm{LST} = 2\overline{\rho}\frac{\partial \overline{U}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}, \tag{2}$$

and turbulence-turbulence (TT) interaction (also denoted NST for "non-linear source term"):

$$\mathrm{TT} \equiv \mathrm{NST} = \frac{\partial^2}{\partial x_i \partial x_j}[\overline{\rho}(u'_i u'_j - \overline{u'_i u'_j})]. \tag{3}$$

In this model, LST and NST are modeled using data (experimental or DNS), and an expression for the PSD is derived analytically from (1) given a (modeled) right-hand-side. this is accomplished by first transforming into Fourier space (taking the Fourier transform of (1) in the x_1– and x_3–coordinate directions), and then using the Green's function for the resulting ordinary differential equation (ODE) to write an expression for the solution in Fourier space [1, 2, 4, 5]. The details of this derivation are omitted here but can be found in [5].

Consider the canonical problem of an incompressible, turbulent boundary layer over a flat plate, located in the $z \equiv x_3 = 0$ plane. Eqn. (1) will be simplified for this problem and an analytical expression for the PSD will be derived from this equation. The following physical assumptions are made:

- Standard boundary layer assumptions for an incompressible turbulent boundary layer (see Section 7.3 of [9]).
- Flow in planes parallel to the wall is homogeneous and stationary[1].
- Only the mean flow gradient normal to the wall is important.
- The turbulence field is frozen and convects with a velocity U_c.

From the third assumption above, it follows that (2) reduces to:

$$\mathrm{LST} = -2\overline{\rho}\frac{\partial \overline{U}}{\partial y}\frac{\partial v'}{\partial x}, \tag{4}$$

where $\overline{U} \equiv \overline{U}_1$, $v' \equiv u'_2$, $x \equiv x_1$ and $y \equiv x_2$. The fourth assumption above is justified by the observation that the intent is to predict the frequency spectrum of wall-pressure fluctuations, which are dominantly generated by convected turbulence activity [1, 2].

The question of the relative importance/unimportance of NST (3) must now be addressed. Some discussion on this matter can be found in [1, 2, 8]. Near wall profiles of NST are shown in [8] by Kim, who ran a DNS to study channel flow. Kim's results suggest that:

- The ratio of the NST to the LST is "universal" in a region close to the wall in turbulent mean flows.

Thus, for a turbulent boundary layer, (1) can be simplified as:

$$\Delta p' = -2\overline{\rho}\frac{\partial \overline{U}}{\partial y}\frac{\partial v'}{\partial x}\left[1 + \frac{\mathrm{NST}}{\max(\mathrm{LST})}\right]. \tag{5}$$

[1] This assumption is what allows taking the Fourier transform of (1) in the x_1 and x_3 coordinate directions.

The terms in the brackets in (5) are to be modeled. They are a function of $y^+ \equiv y/\delta_\nu$ only [1, 2], where $\delta_\nu \equiv \frac{\nu}{u_\tau}$, and $u_\tau \equiv \sqrt{\tau_w/\overline{\rho}}$.

Assuming that the

- Normal pressure gradient is zero at the wall,

it can be shown following the derivation of [5] that the (analytical) expression for the pressure power spectral density (PSD) based on (5) is:

$$\boxed{\Phi_{pp}(\omega)\int_0^\delta\int_0^\delta \frac{d\overline{U}(y)}{dy}\frac{d\overline{U}(y')}{dy'}\overline{\rho}(\overline{v'^2(y)})^{1/2}(\overline{v'^2(y')})^{1/2}\left[1+\frac{\mathrm{NST}}{\max(\mathrm{LST})}\right]\mathscr{F}(y,y',\omega)dydy'.} \tag{6}$$

Here, the conventional assumption that the k_1 dependence in the spectrum can be replaced by the frequency ω,

$$\omega = k_1 U_c, \tag{7}$$

has been employed, where U_c is the velocity of the turbulence field, and k_1 is the wave number that comes from taking the Fourier transform of (1) in the x_1 coordinate direction. The function $\mathscr{F}$ that appears in (6) is given by:

$$\mathscr{F}(y,y',\omega) = \int_{-\infty}^{\infty}\left(\frac{k_c^2}{k_c^2+k_3^2}\right)e^{-\sqrt{k_c^2+k_3^2}(y+y')}\phi_{22}(y,y',k_3,\omega/U_c)dk_3, \tag{8}$$

where $k_c \equiv \omega/U_c$ is the convective wave number, and ϕ_{22} is the spectral correlation function for the vertical fluctuation velocities between two positions y and y' within the boundary layer.

3 Required Inputs and Models

The following inputs (models) are required to evaluate (6):

- The gradient of the mean streamwize velocity profile as a function of y (coordinate across the boundary layer): $\frac{d\overline{U}(y)}{dy}$. This would be computed in a RANS, or estimated from classical boundary layer theory.

- A model for $(\overline{v'^2(y)})^{1/2}$ (the root-mean-square (rms) y–velocity fluctuation inside the boundary layer as a function of y). This could be obtained from a RANS (e.g., the k equation of the $k-\varepsilon$ model). Note that the model for this quantity may depend on whether the turbulence is isotropic or anisotropic. Corrections for an anisotropic turbulence field are discussed in [1, 2].

- A value/model for U_c, the convection velocity. Some discussion of the convection velocity can be found in [6], where it is shown, for example that convection velocities for a smooth wall asymptote to a constant as y increases, and can be as high as $0.8U_\infty$.

- A model for $\frac{\mathrm{NST}}{\max(\mathrm{LST})}$, if desired (i.e., if NST is thought to be nonzero).

- A model for ϕ_{22}, the spectral vertical fluctuation velocity fluctuation between two positions y and y'. Alternatively, this expression can be written as [1, 2]

$$\phi_{22}(y,y',k_3,\omega/U_c) = E_{22}^N(y,y',k_3,\omega/U_c)\sqrt{\phi_1(\omega/U_c,y)\phi_1(\omega/U_c,y')}, \tag{9}$$

  where E_{22}^N is the interplane correlation function between y and y' and ϕ_1 is the autospectrum of v'. Lee *et. al.* model the latter quantity using Farabee's velocity data [3] as

$$\phi_1(k_c,y) = \frac{y_{pp}}{U_c}\frac{0.18}{1+\left(\frac{k_c y_{pp}}{1.7}\right)^2}, \tag{10}$$

$$\frac{y_{pp}}{\delta^*} = \frac{\frac{1.5y}{\delta^*}\left(0.9+\frac{2.5y}{\delta^*}\right)}{1+\left(\frac{2.5y}{\delta^*}\right)^2}. \tag{11}$$

According to Lee *et. al.*, E_N is modeled according to "Chase's formulation [7]". It is unclear if the expression for E_{22}^N employed by Lee *et. al* is actually given in [7], however. The paper appears to have explicit models for E_{11}^N only (p. 43, Fig. 3 in [7]). Likely, Lee *et. al* used data fitting procedures similar to those employed by Chase to obtain a model for E_{22}^N.

- The density $\overline{\rho}$ and an estimate of δ, the boundary layer thickness. If the model ϕ_1 (10) is used, δ^* is needed as well. Note that for any model of ϕ_{22} (9) δ^* will likely be an implicit or explicit input (that is, ϕ_{22} will likely depend on δ^*). If y, $\overline{U}$, etc. are to be non-dimensionalized in (6), τ_w and ν will be required as well.

4 Discussion

As mentioned above, the derivation of the pressure PSD (6) is based on the incompressible pressure-Poisson equation (1). The equation for p' that arises by starting with the compressible equations is fundamentally different: it is not a Poisson equation but rather a convective wave equation (e.g., Eqn. (6) in [12]). The Green's function for this equation is substantially different and more complex than the Green's function for (1) so an analytical derivation of Φ_{pp} such as the one described above may not be possible in general starting from compressible equations. However, it would be worthwhile to study the validity of (6) in a compressible regime to see if it is a reasonable model despite having been derived from incompressible equations.

Another assumption inherent in the derivation above is that the flow is of an equilibrium turbulent boundary layer (zero streamwise pressure gradient and turbulence characteristics possessing in-plane homogeneity). Lee *et. al.* have actually applied the model, with success, to non-equilibrium flows, namely a backward-facing step (BFS) flow problem [1, 2], which contains a recirculation region and a reattachment region in which vortical structures are generated. Success of the model on this problem suggests the model may be applicable to more complex flow scenarios than the one from which it was derived.

Given models for the unclosed terms (listed above), it is straightforward to evaluate (6) (e.g., using numerical integration in MATLAB), thereby generating a pressure PSD. The difficulty comes in closing these terms, in particular the interplane correlation E_{22}^N, which appears in (9). It is unclear from [1, 2] and the references given in these work exactly what expression (model) was used for this function. Without this expression, one cannot expect to reproduce the results in [1, 2] (as a first step, and for debugging purposes). An alternative would be to use alternate data for this quantity (e.g., tunnel data), chosen to be consistent with a particular selected flow scenario for which pressure PSD data consistent with models for all the unclosed terms are available to compare to.

As mentioned in the Introduction, the model discussed herein does not give the probability density function (PDF) for the pressure fluctuations, which is ultimately required for the application of interest, in addition to the PSD .

References

[1] Y.-T. Lee, T.M. Farabee, W.K. Blake. Turbulence effects of wall-pressure fluctuations for reattached flow. *Comput. & Fluids* **38** (1009) 1033–1041.

[2] Y.-T. Lee, W.K. Blake, T.M. Farabee. Modeling of wall pressure fluctuations based on time mean flow field. *J. Fluids Engng.* **127** (2005) 233–240.

[3] T.M. Farabee, M.J. Casarella. Spectral features of wall pressure fluctuations beneath turbulent boundary layers. *Phys. Fluids A* (1991) 3–10.

[4] R.L. Panton, J.H. Linebarger. Wall pressure spectral calculations for equilibrium boundary layers. *J. Fluid Mech.* **65**(2) (1974) 261–287.

[5] R.H. Kraichnan. Pressure fluctuations in turbulent flow over a flat plate. *J. Acoust. Soc. of Americal* **28** (3) (1956) 378–390.

[6] W.K. Blake. Turbulent boundary-layer wall-pressure fluctuations on smooth and rough walls. *J. Fluid Mech.* **44** (4) (1970) 637–660.

[7] D.M. Chase. Modeling the wavevector-frequency spectrum of turbulent boundary layer wall pressure. *J. Sound & Vibration* **70** (1) (1980) 29–67.

[8] J. Kim. On the structure of pressure fluctuations in simulated turbulent channel flow. *J. Fluid. Mech.* **205** (1989) 412–451.

[9] S.B. Pope. Turbulent Flows. New York, NY: Cambridge University Press, 2000.

[10] L.J. DeChant. A semi-infinite domain eigenvalue problem describing turbulent velocity fluctuations. *Comput. & Math. with Appl.* **60** (5) (2010) 1177–1221.

[11] S.J. Beresh, J.F. Henfling, R.W. Spillers, B.O.M. Pruett. *40th AIAA Fluid Dynamics Conference*, AIAA Paper 2010–4274, Chicago, IL (2010).

[12] W.D. Thacker, S. Sarkar, T.B. Gatski. Analyzing the influence of compressibility on the rapid pressure-strain rate correlation in turbulent shear flows. *Theor. Comput. Fluid Dyn.* **21** (2007) 171–199.

[13] L.J. DeChant, J.L. Payne. A convection reaction diffusion (CRD) equation model for 2-d transition pressure fluctuation behavior. *29th AIAA Aeroacoustics Conference*, AIAA 2008–3020. Vancouver, British Columbia, Canada (2008).

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