The Discontinuous Enrichment Method for Multi-Scale Fluid Problems

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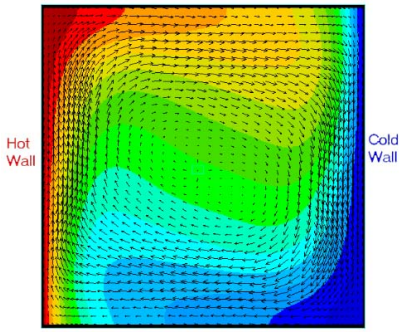
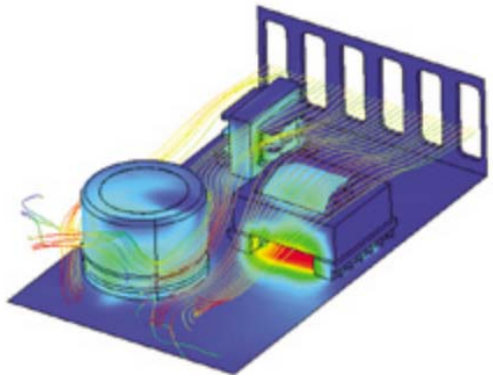
International Congress on Industrial & Applied Mathematics
(ICIAM 2011)
Friday, July 22, 2011

2D Scalar Advection-Diffusion Equation

$$\mathcal{L}c = \underbrace{-\kappa\Delta c}_{\text{diffusion}} + \underbrace{\mathbf{a}\cdot\nabla c}_{\text{advection}} = f$$

- Advection velocity: $\mathbf{a} = (a_1, a_2)^T = |\mathbf{a}|(\cos\phi, \sin\phi)^T$.
- ϕ = advection direction.
- κ = diffusivity.





- Describes many transport phenomena in fluid mechanics:
  - Heat transfer.
  - Semi-conductor device modeling.
  - Usual scalar model for the more challenging Navier-Stokes equations.

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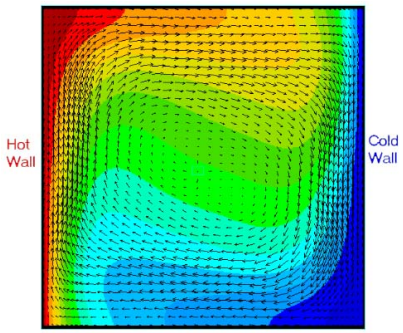

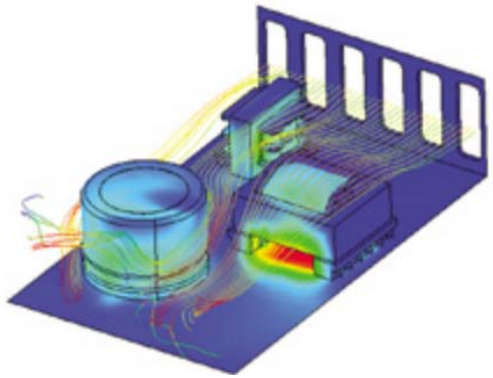

- Describes many transport phenomena in fluid mechanics:
  - Heat transfer.
  - Semi-conductor device modeling.
  - Usual scalar model for the more challenging Navier-Stokes equations.
- Global **Péclet number** (L = length scale associated with Ω):

$$Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{L|\mathbf{a}|}{\kappa} = Re \cdot \begin{cases} Pr & \text{(thermal diffusion)} \\ Sc & \text{(mass diffusion)} \end{cases}$$

Advection-Dominated Regime

- Typical applications: flow is advection dominated.

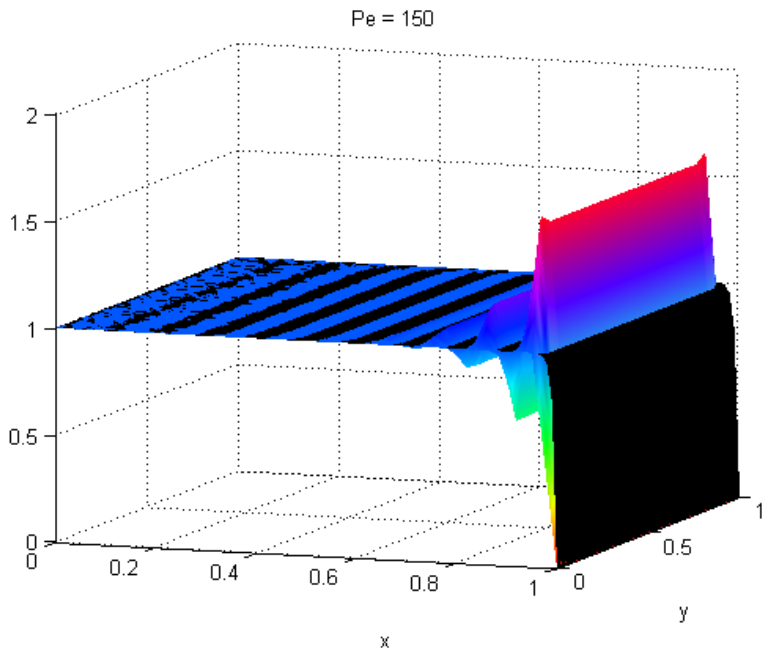

Figure 1: Galerkin Q_1 solution (color) vs. exact solution (black) for $Pe = 150$

Advection-Dominated (High Pe) Regime

$\Downarrow$

Sharp gradients in exact solution

$\Downarrow$

Galerkin FEM inadequate: spurious oscillations (Fig. 1)

- Objective of DEM:
  - To produce efficiently an oscillation-free solution.
  - Alternative to **stabilized FEMs** (SUPG, GLS, USFEM).

The Discontinuous Enrichment Method (DEM)

- First developed by Farhat *et al.* in 2000 [1] for the Helmholtz equation.

Idea of DEM:

Employ as the finite element shape functions the free-space solutions to the governing homogeneous constant-coefficient PDE $\mathcal{L}c = 0$.

$$c^h = c^E \in \mathcal{V}^E$$

where

$$\mathcal{V}^E = \text{span}\{c : \mathcal{L}c = 0\}$$

- Continuity across element boundaries is enforced weakly using Lagrange multipliers $\lambda^h \in \mathcal{W}^h$:

$$\lambda^h \approx \nabla c_e^E \cdot \mathbf{n}^e = -\nabla c_{e'}^E \cdot \mathbf{n}^{e'} \qquad \text{on } \Gamma^{e,e'}$$

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DEM = DGM with Lagrange Multipliers

Discretization, Implementation & Computational Complexity

- Element matrix problem (uncondensed):

$$\begin{pmatrix} \mathbf{k}^{\mathrm{EE}} & \mathbf{k}^{\mathrm{EC}} \\ \mathbf{k}^{\mathrm{CE}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}^{\mathrm{E}} \\ \boldsymbol{\lambda}^{h} \end{pmatrix} = \begin{pmatrix} \mathbf{r}^{\mathrm{E}} \\ \mathbf{r}^{\mathrm{C}} \end{pmatrix}$$

Due to the discontinuous nature of $\mathcal{V}^{E}$, $\mathbf{c}^{\mathrm{E}}$ can be eliminated at the element level by a static condensation

- Statically-condensed matrix problem:

$$-\mathbf{k}^{\mathrm{CE}}(\mathbf{k}^{\mathrm{EE}})^{-1}\mathbf{k}^{\mathrm{EC}}\boldsymbol{\lambda}^{h} = \mathbf{r}^{C} - \mathbf{k}^{\mathrm{CE}}(\mathbf{k}^{\mathrm{EE}})^{-1}\mathbf{r}^{\mathrm{E}}$$

Static condensation greatly reduces computational complexity!

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Static condensation greatly reduces computational complexity!

  - Complexity depends on number of Lagrange multiplier approximations per edge.
  - Sparser global system than FEM.

Polynomial Enrichment Functions for 2D Advection-Diffusion

- Polynomial free-space solutions to $\mathbf{a} \cdot \nabla c_n^E - \Delta c_n^E = 0$ (of any desired degree n) can be derived.

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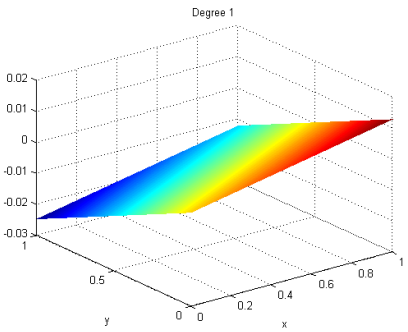
$$c_1^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|$$



$c_1^E(\mathbf{x})$

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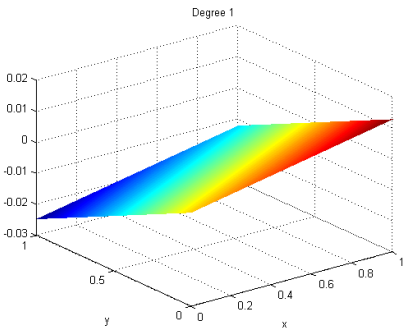
$$c_1^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|$$

$$c_2^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|^2 + 2(\mathbf{a} \cdot \mathbf{x})$$



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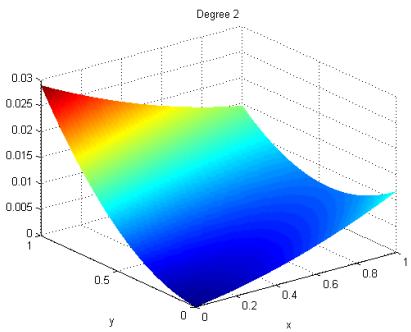

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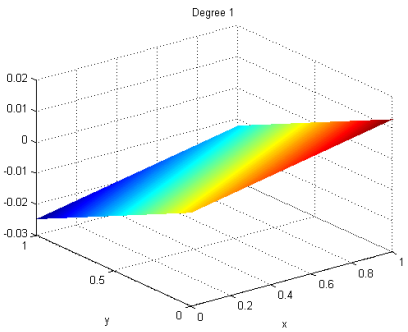
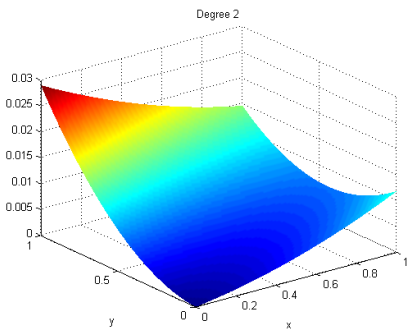
$$c_2^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|^2 + 2(\mathbf{a} \cdot \mathbf{x})$$

$$c_3^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|^3 + 6|\mathbf{a} \times \mathbf{x}|(\mathbf{a} \cdot \mathbf{x})$$

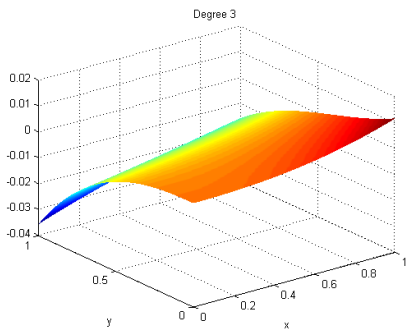
$$\vdots$$



$c_1^E(\mathbf{x})$



$c_2^E(\mathbf{x})$



$c_3^E(\mathbf{x})$

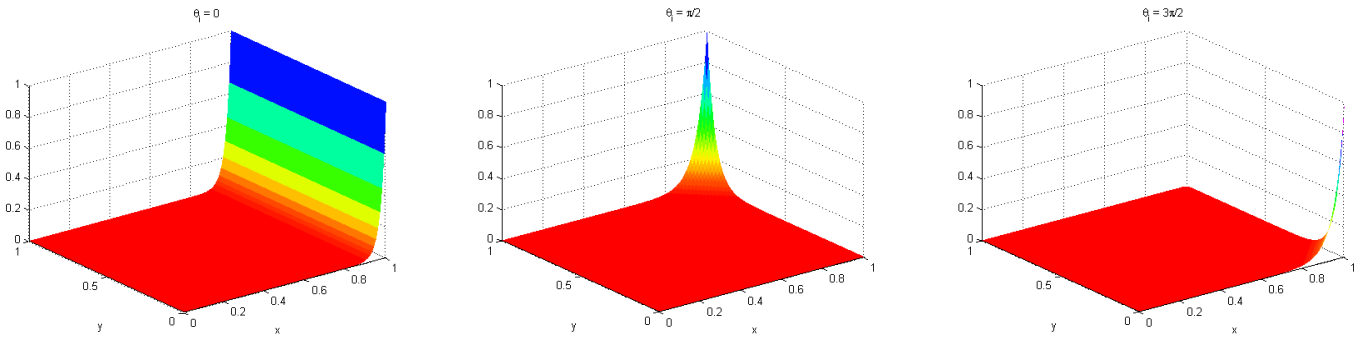
Slowly-varying (coarse) scale shape functions

Angle-Parametrized Exponential Enrichment Functions for 2D Advection-Diffusion

- Exponential free-space solutions to $\mathbf{a} \cdot \nabla c^E - \kappa \Delta c^E = 0$ can be derived as well.

$$\boxed{c^E(\mathbf{x}; \theta_i) = e^{\left(\frac{a_1 + |\mathbf{a}| \cos\theta_i}{2\kappa}\right)(x - x_{r,i})} e^{\left(\frac{a_2 + |\mathbf{a}| \sin\theta_i}{2\kappa}\right)(y - y_{r,i})}} \tag{1}$$

$$\Theta^c \equiv \{\theta_i\}_{i=1}^{n^E} \in [0, 2\pi) = \text{ set of angles specifying } \mathcal{V}^E$$



$\phi = 0, \theta_i = 0$ $\phi = 0, \theta_i = \frac{\pi}{2}$ $\phi = 0, \theta_i = \frac{3\pi}{2}$

Figure 2: Plots of enrichment functions $c^E(\mathbf{x}; \theta_i)$ for several values of θ_i ($Pe = 20$)

Rapidly-varying (fine) scale shape functions

Homogeneous Boundary Layer Problem

- $\Omega = (0,1) \times (0,1)$, $f = 0$.
- $\mathbf{a} = \begin{pmatrix} \cos\phi, & \sin\phi \end{pmatrix}$.
- Dirichlet boundary conditions are specified on Γ such that the exact solution to the BVP is given by

$$c_{ex}(\mathbf{x};\phi,\psi) = \frac{e^{\frac{1}{2\kappa}\{[\cos\phi+\cos\psi](x-1)+[\sin\phi+\sin\psi](y-1)\}} - 1}{e^{-\frac{1}{2\kappa}[\cos\phi+\cos\psi+\sin\phi+\sin\psi]} - 1}$$

- $\psi \in [0, 2\pi)$: some flow direction (not necessarily aligned with ϕ).
- Solution exhibits a sharp exponential boundary layer in the advection direction ϕ, whose gradient is a function of the Péclet number.

Figure 3: $\phi = \psi = 0$

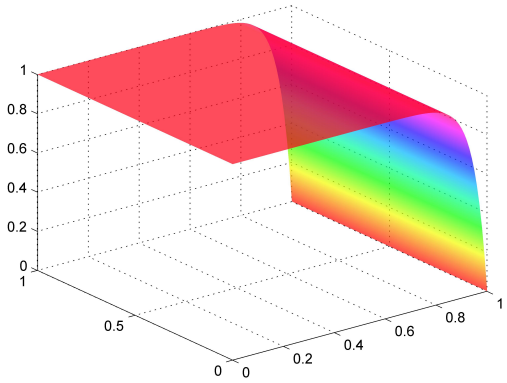
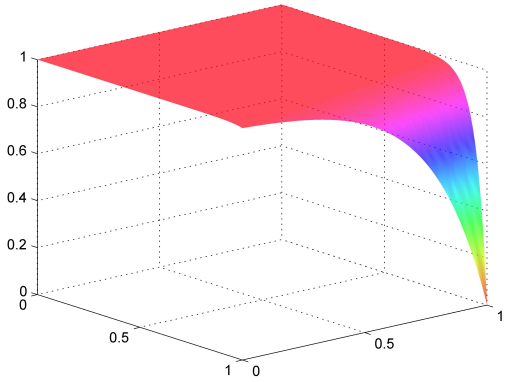
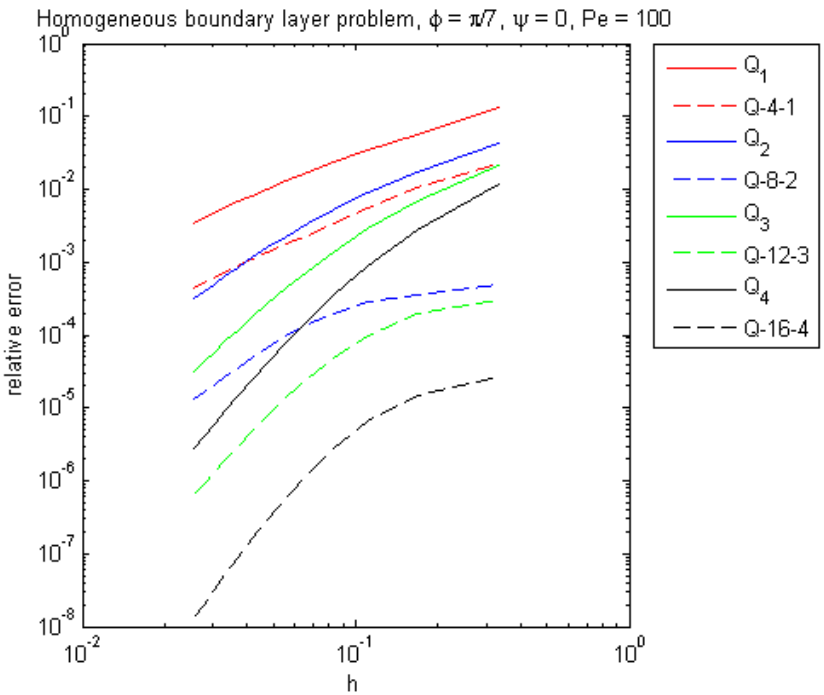


Figure 4: $\phi = \pi/7, \psi = 0$



Convergence Analysis & Results ($\phi = \pi/7, \psi = 0$)



Element	Rate of convergence	# dofs to achieve 10^{-3} error
Q_1	1.90	63,266
Q-4-1	1.99	14,320
Q_2	2.38	24,300
Q-8-2	3.27	5400
Q_3	3.48	12,500
Q-12-3	3.88	850
Q_4	4.41	8600
Q-16-4	5.19	570

- To achieve for this problem the relative error of 0.1% for $Pe = 10^3$:
  - Q-4-1 and Q-8-2 require $\approx 4.5 \times$ **fewer** dofs than Q_1 and Q_2 respectively.
  - Q-12-3 and Q-16-4 require $\approx 15 \times$ **fewer** dofs than Q_3 and Q_4 respectively.

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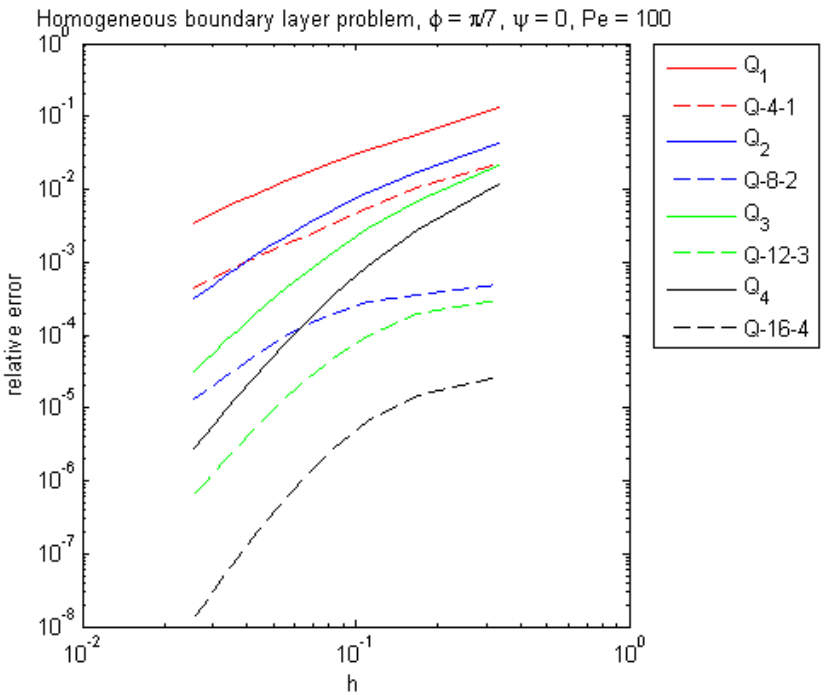

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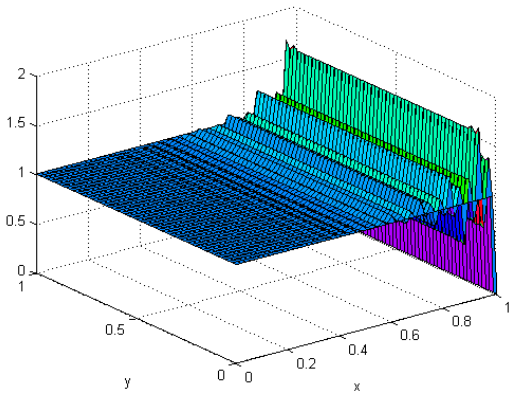
  - Q-4-1 and Q-8-2 require $\approx 4.5 \times$ **fewer** dofs than Q_1 and Q_2 respectively.

    $\Rightarrow 8 \times$ less CPU time.
  - Q-12-3 and Q-16-4 require $\approx 15 \times$ **fewer** dofs than Q_3 and Q_4 respectively.

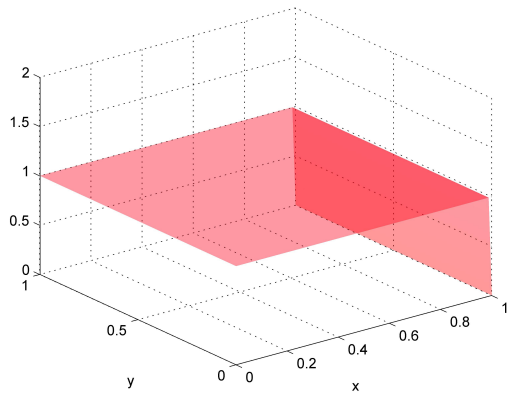
    $\Rightarrow 40 \times$ less CPU time.

Solution Plots for Homogeneous BVP

Figure 5: $\phi = \psi = 0$, $Pe = 10^3$, $\approx$ 1600 dofs

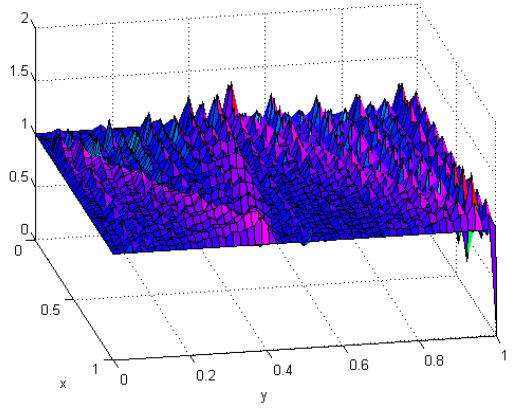


Q_3

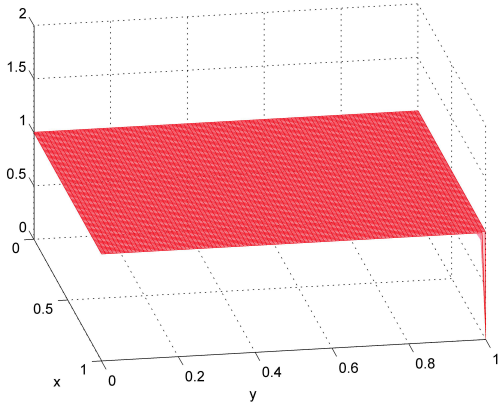


Q-12-3

Figure 6: $\phi = \pi/7$, $\psi = 0$, $Pe = 10^5$, $\approx$ 1600 dofs



Q_3

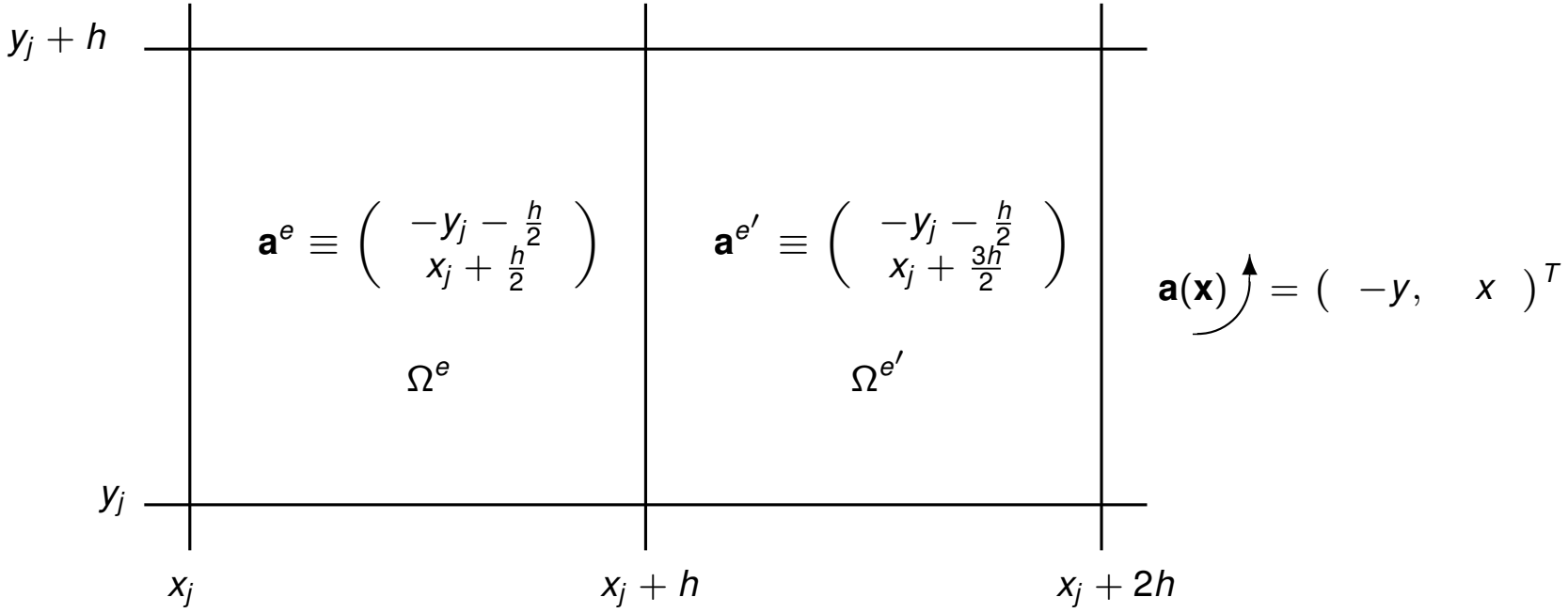


Q-12-3

Extension to Variable-Coefficient Problems

- Define $\mathcal{V}^E$ *within each element* as the free-space solutions to the homogeneous PDE, with locally-*frozen* coefficients.
- $\mathbf{a}(\mathbf{x}) \approx \mathbf{a}^e$ =constant inside each element Ω^e as $h \to 0$:

$$\{\mathbf{a}(\mathbf{x}) \cdot \nabla c - \kappa \Delta c = f(\mathbf{x}) \ \text{ in } \ \Omega\} \approx \cup_{e=1}^{n_{el}} \{\mathbf{a}^e \cdot \nabla c - \kappa \Delta c = f(\mathbf{x}) \ \text{ in } \ \Omega^e\}.$$



- Enrichment in each element:

$$c_e^E(\mathbf{x}; \theta_i^e) = e^{\frac{|\mathbf{a}^e|}{2\kappa}(\cos\phi^e + \cos\theta_i^e)(x - x_{r,i}^e)} e^{\frac{|\mathbf{a}^e|}{2\kappa}(\sin\phi^e + \sin\theta_i^e)(y - y_{r,i}^e)} \in \mathcal{V}_e^E$$

"Higher Order" Enrichment Function for Variable-Coefficient Advection-Diffusion

- Linearize $\mathbf{a}(\mathbf{x})$ to second order, instead of to first order:

$$\mathbf{a}(\mathbf{x}) \approx \mathbf{a}(\bar{\mathbf{x}}^e) + \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \cdot (\mathbf{x} - \bar{\mathbf{x}}^e) \qquad \text{in } \Omega^e$$

- Enrich with free-space solutions to

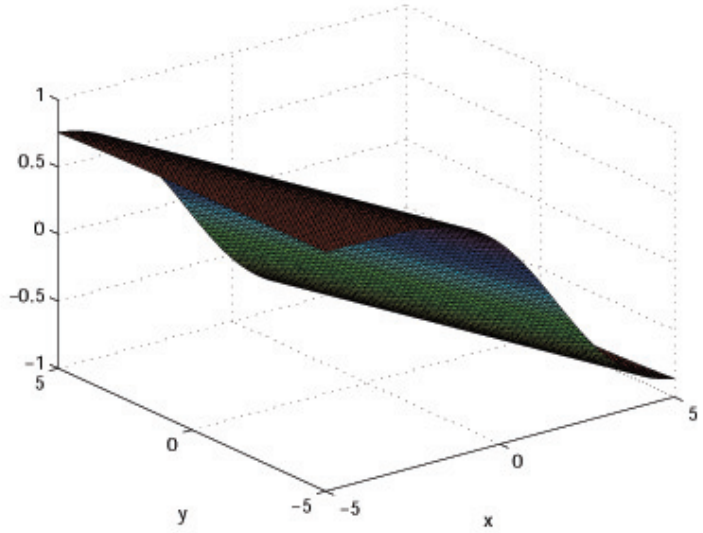
$$[\mathbf{A}\mathbf{x} + \mathbf{b}] \cdot \nabla c^E - \kappa \Delta c^E = 0 \tag{2}$$

where $\mathbf{A} \equiv \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$, $\mathbf{b} \equiv (\mathbf{a}(\bar{\mathbf{x}}^e) - \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \bar{\mathbf{x}}^e)$.

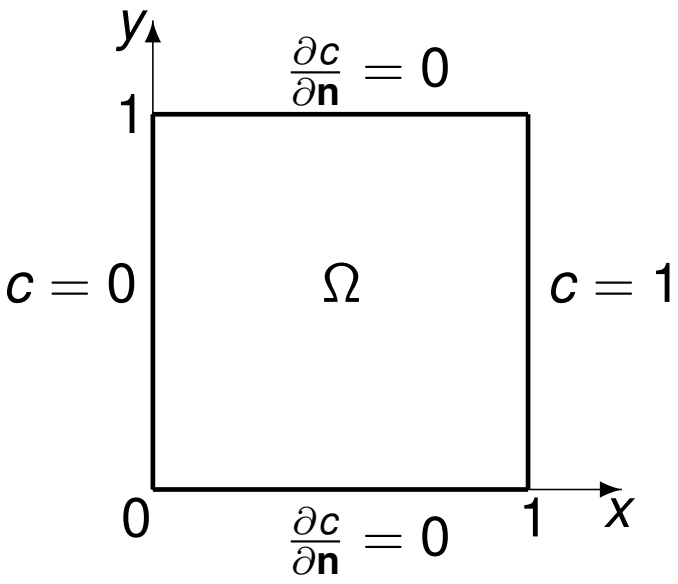
- Solutions to (2) are given by:

$$\boxed{c^E(\mathbf{x}) = \int_0^{\mathbf{v}_i \cdot \mathbf{x}} \exp\left\{ \frac{\sigma_i w^2}{2} + (\mathbf{v}_i \cdot \mathbf{b}) w \right\} dw}$$

σ_i = eigenvalue of $\nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$
$\mathbf{v}_i$ = eigenvector of $\nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$



Lid-Driven Cavity Flow Problem

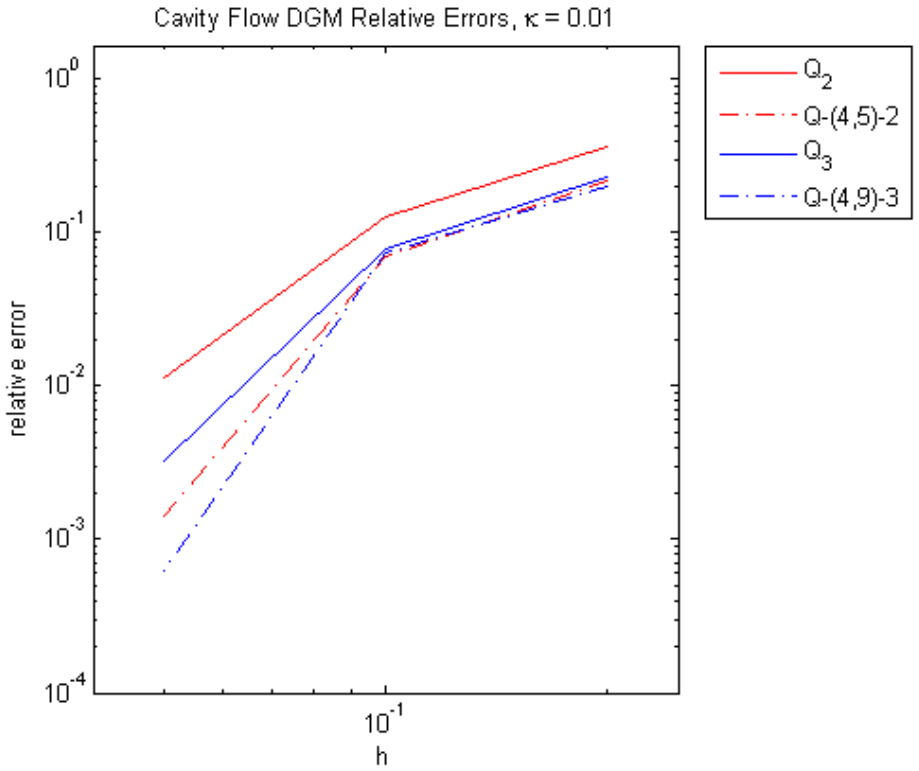


- $\Omega = (0,1) \times (0,1)$, $f = 0$.
- $\mathbf{a}(\mathbf{x})$ computed numerically by solving the incompressible Navier-Stokes equations for lid-driven cavity flow problem (stationary sides and bottom, tangential movement of top).
- Advection field reconstructed using interpolation with bilinear shape functions ϕ_i^e:

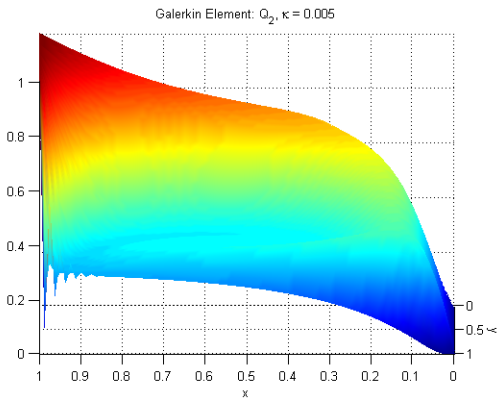
$$\mathbf{a}^e(\xi) = \sum_{i=1}^{\#\text{ nodes of }\Omega^e} \mathbf{a}_i^e \phi_i^e(\xi)$$

- $c(\mathbf{x})$ represents temperature in cavity.

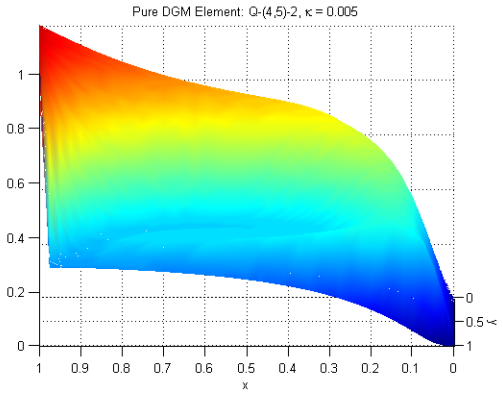
Convergence Analysis & Results ($\kappa = 0.01$, $Pe \approx 260$)



- DEM elements without "higher order" enrichment outperform Galerkin comparables.

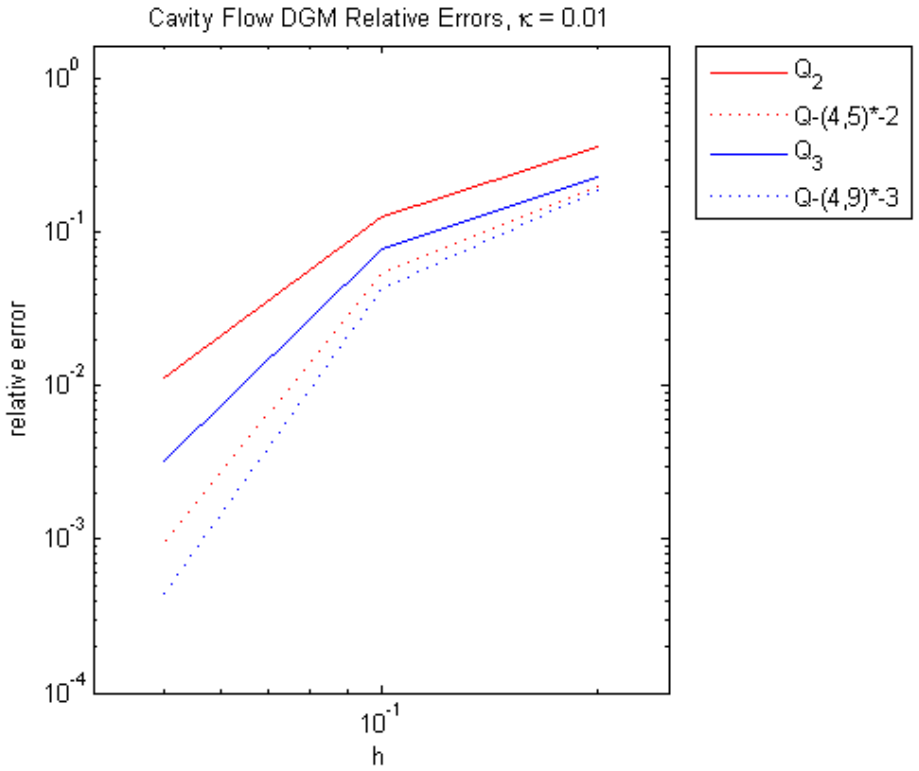


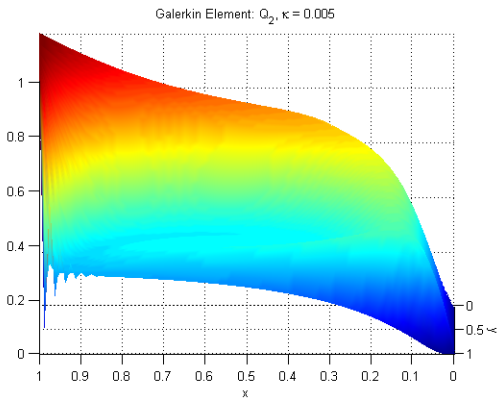
Q_2



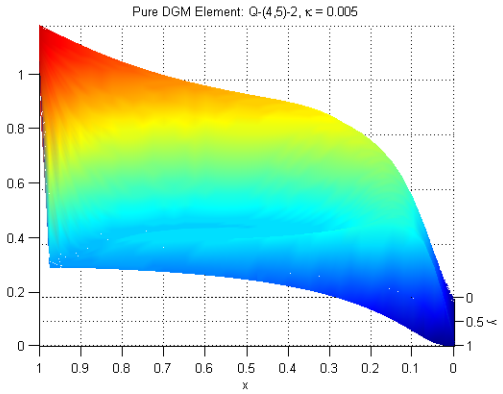
$Q - (4, 5) - 2$

Convergence Analysis & Results ($\kappa = 0.01$, $Pe \approx 260$)





Q_2



$Q - (4, 5) - 2$

- DEM elements without "higher order" enrichment outperform Galerkin comparables.
- Further improvement in computation by adding "higher order" enrichment.

DEM for the Unsteady Advection-Diffusion Equation

- Unsteady advection-diffusion equation:

$$c_t + \mathbf{a}(\mathbf{x}, t) \cdot \nabla c - \kappa \Delta c = 0$$

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$$\frac{c^{n+1} - c^n}{\Delta t} + \mathbf{a}^n(\mathbf{x}) \cdot \nabla c^{n+1} - \kappa \Delta c^{n+1} = 0$$

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- Enrichment functions inside each element at time step n are the free-space solutions to steady version of the equation above:

$$\boxed{\mathcal{V}_e^{E,n} = \text{span}\{c^n(\mathbf{x}) : \mathbf{a}^{n-1}(\bar{\mathbf{x}}_e) \cdot \nabla c^n - \kappa \Delta c^n = 0, \mathbf{x} \in \Omega^e\}}$$

where

$$\mathcal{V}_e^{E,n} = \text{enrichment field inside element } \Omega^e \text{ at time step } n$$

$$\bar{\mathbf{x}}_e \equiv \text{midpoint of element } \Omega^e$$

Natural Convection in a Differentially-Heated Cavity

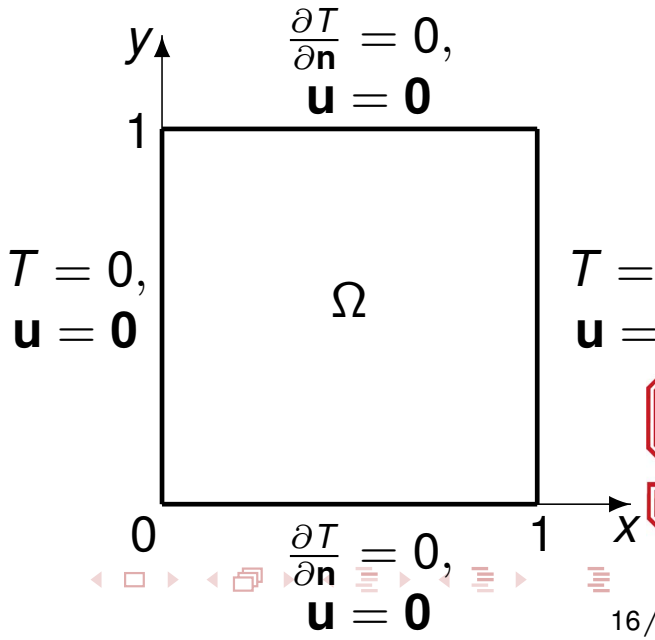
- Incompressible Navier-Stokes equations with Boussinesq temperature approximation.

$$\begin{cases} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{Gr^{0.5}} \Delta \mathbf{u} &= -\nabla p + T \mathbf{e}_2 \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \frac{1}{PrGr^{0.5}} \Delta T &= 0 \end{cases}$$

where

$$\begin{aligned} \mathbf{u}^T = (u_1(\mathbf{x}, t), \quad u_2(\mathbf{x}, t)) : &\quad \text{fluid velocity vector} \\ p = p(\mathbf{x}, t) : &\quad \text{fluid pressure} \\ T = T(\mathbf{x}, t) : &\quad \text{fluid temperature} \end{aligned}$$

- $\Omega = (0, 1)^2$.
- No-slip boundary conditions on $\mathbf{u}$ on sides of box.
- At time $t = 0$ begin to heat right wall; top walls of box are insulating (adiabatic).



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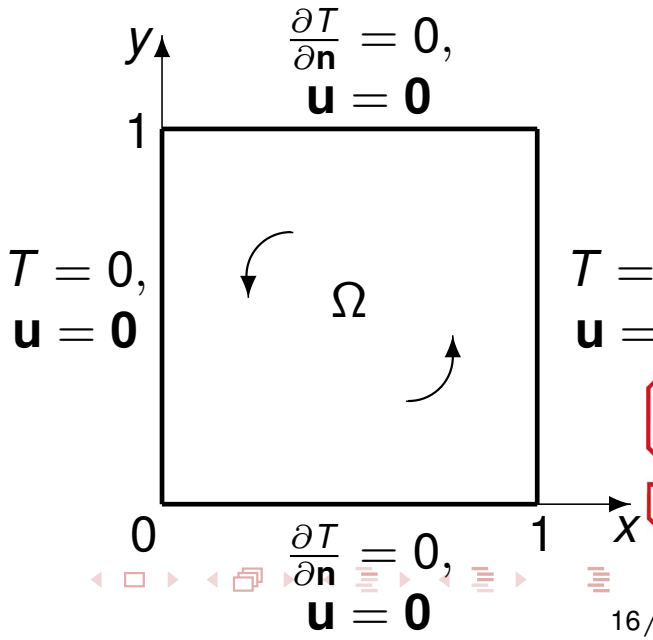
Temperature gradient induces counterclockwise flow field



Simulation: Galerkin T vs. DGM T ($Ra = Gr = 1000$)

u, v : Galerkin Q_3
p : Galerkin Q_2

T : Galerkin Q_1

T : DGM Q-4-1

Summary

Discontinuous Enrichment Method (DEM) = efficient, competitive alternative to stabilized FEMs for advection-dominated transport problems in CFD.

- Parametrization of exponential basis enables systematic design of DEM elements of arbitrary orders.
- Augmentation of enrichment space with additional free-space solutions can improve further the approximation.
- For all test problems, enriched elements outperform their Galerkin and stabilized Galerkin counterparts of comparable computational complexity, sometimes by many orders of magnitude.
- In a high Péclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.
- Advection-diffusion work generalizable to more complex equations in fluid mechanics (e.g., non-linear, 3D).
- Future work: DEM for incompressible Navier-Stokes.

References (www.stanford.edu/∼irinak/pubs.html)

[1] C. Farhat, I. Harari, L.P. Franca. The Discontinuous Enrichment Method. *Comput. Meth. Appl. Mech. Engng.* **190** (2001) 6455-6479.

[2] **I. Kalashnikova**, R. Tezaur, C. Farhat. A Discontinuous Enrichment Method for Variable Coefficient Advection-Diffusion at High Peclet Number. *Int. J. Numer. Meth. Engng.* **87** (2010) 309–335.

[3] C. Farhat, **I. Kalashnikova**, R. Tezaur. A Higher-Order Discontinuous Enrichment Method for the Solution of High Peclet Advection-Diffusion Problems on Unstructured Meshes. *Int. J. Numer. Meth. Engng.* **81** (2010) 604-636.

[4] **I. Kalashnikova**, C. Farhat, R. Tezaur. A Discontinuous Enrichment Method for the Solution of Advection-Diffusion Problems in high Peclet Number Regimes. *Fin. El. Anal. Des.* **45** (2009) 238-250.

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Thank you! Questions?

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Appendix: Computational Complexities

Element	Asymptotic # of dofs	Stencil width for uniform $n \times n$ mesh	(# dofs) $\times$ (stencil width)	L^2 convergence rate (*a posteriori*)
Q_1	n_{el}	9	$9n_{el}$	2
$Q-4-1$	$2n_{el}$	7	$14n_{el}$	2
Q_2	$3n_{el}$	21	$63n_{el}$	3
DGM with $n^\lambda = 2$	$4n_{el}$	14	$56n_{el}$	3
DEM with $n^\lambda = 1$	$3n_{el}$	21	$63n_{el}$	$2-3$
Q_3	$5n_{el}$	33	$165n_{el}$	4
DGM with $n^\lambda = 3$	$6n_{el}$	21	$126n_{el}$	4
DEM with $n^\lambda = 3$	$5n_{el}$	33	$165n_{el}$	$3-4$
Q_4	$7n_{el}$	45	$315n_{el}$	5
DEM with $n^\lambda = 3$	$7n_{el}$	45	$315n_{el}$	$4-5$

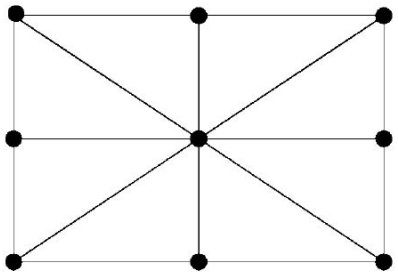

Figure 7: Q_1 stencil

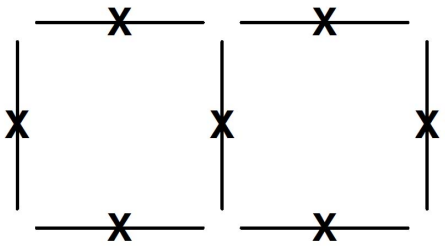

Figure 8: 1st order DGM stencil