Structure Preserving Eigensolvers

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Goals

- Solve complete eigenproblem $Ax = \lambda x$
  - A is real and doubly-structured:

symmetric or *skew-symmetric*

and

persymmetric or *perskew-symmetric*

- Algorithms
  - Preserve both structures
  - Good convergence behavior
  - Parallelizable
  - Stable, accurate

Structures

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

Persymmetric

$p\mathcal{S}(n) = \{S \in \mathbb{R}^{n\times n} : (RS)^T = RS\}$

Perskew-symmetric

$p\mathcal{K}(n) = \{S \in \mathbb{R}^{n\times n} : (RS)^T = -RS\}$

Perplectic Orthogonal

$\mathcal{PO}(n) = \{P \in \mathcal{O}(n) : \; RP = PR\}$

$\Longrightarrow$ Preserves double structure

Jacobi Method (1846)

Goal

$$\begin{bmatrix} & & \\ & real, \quad symmetric & \\ & & \end{bmatrix}_{n\times n} \Rightarrow \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}_{n\times n}$$

Idea

Repeat until convergence

1. Choose 2×2 structured subproblem
2. Diagonalize it

End Repeat

Our Method

2×2 Subproblem Insufficient: $\mathcal{PO}(2) = \{I, -I, R, -R\}$

- Choose 4×4 Subproblem

  Smaller problem must have same double structure

  Directly put 4×4 matrix in canonical form

- Using $\mathbb{R}^{4\times 4} \cong \mathbb{H} \otimes \mathbb{H}$ Translate Problem and Solve

  Quaternions:

  $\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$

- Translate Back to $\mathbb{R}^{4\times 4}$

  Embed 4×4 solution into I_n

  Perform similarity transformation

- Continue Until Convergence

  Will converge if subproblems chosen properly

Canonical Forms

$\begin{bmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_2 & a_2 & 0 \\ b_1 & 0 & 0 & a_1 \end{bmatrix}$ Symmetric Persymmetric	$\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & -b & 0 & 0 \\ -a & 0 & 0 & 0 \end{bmatrix}$ Skew-symmetric Persymmetric
$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -a \end{bmatrix}$ Symmetric Perskew-symmetric	$\begin{bmatrix} 0 & a & b & 0 \\ -a & 0 & 0 & -b \\ -b & 0 & 0 & -a \\ 0 & b & a & 0 \end{bmatrix}$ Skew-symmetric Perskew-symmetric

Theorem *For any of the doubly-structured* $A \in \mathbb{R}^{n \times n}$ *there exists a perplectic orthogonal* $P \in \mathcal{PO}(n)$ *such that* $P^T AP$ *is in canonical form.*

Subproblem Algorithm

Input: *skew-symmetric persymmetric* matrix $A \in \mathbb{R}^{4\times 4}$

Output: *perplectic orthogonal* matrix $P \in \mathbb{R}^{4\times 4}$ such that PAP^T is in canonical form $\begin{bmatrix} & / & \end{bmatrix}$

Algorithm:

$$u = \begin{bmatrix} -a_{12} & 0 & -\frac{1}{2}\left(a_{14} + a_{23}\right) \end{bmatrix}$$

$$v = \begin{bmatrix} 0 & a_{13} & \frac{1}{2}\left(a_{14} - a_{23}\right) \end{bmatrix}$$

$$d_1 = \|u\|_2 + u_2$$

$$d_2 = \|v\|_2 + v_2$$

$$\alpha = 2\sqrt{\|u\|_2 \, \|v\|_2 \, d_1 \, d_2}$$

$$P = \frac{1}{\alpha} \begin{bmatrix} d_1 & 0 & u_1 & 0 \\ 0 & d_1 & 0 & -u_1 \\ -u_1 & 0 & d_1 & 0 \\ 0 & u_1 & 0 & d_1 \end{bmatrix} \begin{bmatrix} d_2 & v_1 & 0 & 0 \\ -v_1 & d_2 & 0 & 0 \\ 0 & 0 & d_2 & -v_1 \\ 0 & 0 & v_1 & d_2 \end{bmatrix}$$

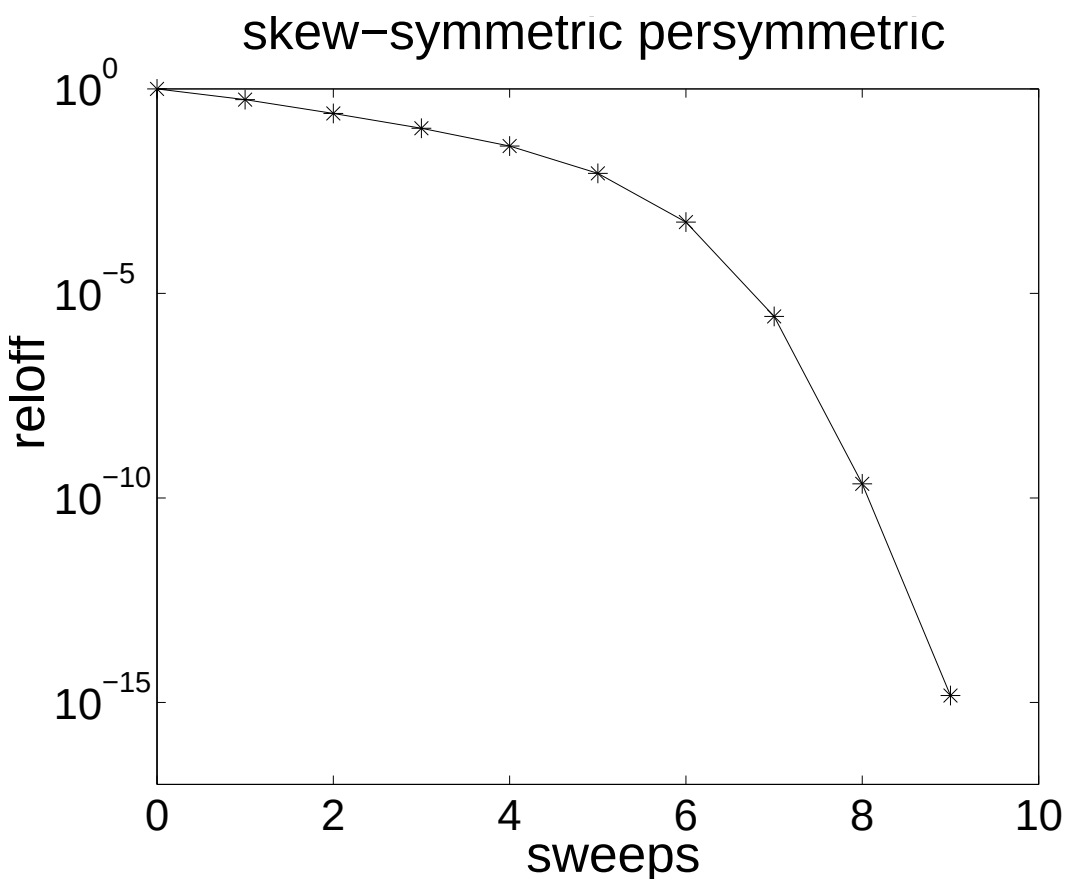
Numerical Results

100 symmetric persymmetric matrices, entries from $N(0,1)$

n	Sweeps	1	2	3	4	5
50	7.27	10^{-16}	10^{-14}	10^{-14}	10^{-15}	10^{-14}
100	8.02	10^{-15}	10^{-13}	10^{-13}	10^{-14}	10^{-13}
150	8.36	10^{-15}	10^{-13}	10^{-13}	10^{-14}	10^{-13}
200	8.85	10^{-15}	10^{-13}	10^{-13}	10^{-14}	10^{-13}
1000	11	10^{-15}	10^{-12}	10^{-12}	10^{-13}	10^{-13}

1. RelOff $= \frac{1}{\|A\|_F}\sqrt{\sum_i \sum_{j\neq i,n-i+1} a_{ij}^2}$
2. $\|P^T RP - R\|_F$ = check of perplectic structure of P
3. $\|P^T P - I\|_F$ = check of orthogonal structure of P
4. Block = check of perplectic orthogonal structure of P
5. RelEig $= \max_j |\lambda_j^{eig} - \lambda_j^{jac}| \,/\, |\lambda_j^{eig}|$

Convergence – 200×200



- Always converged
- Asymptotically quadratic
- RelOff decreased monotonically

Future: We expect these methods to be strongly backward stable and plan to use techniques developed by F. Tisseur (SIMAX 2001) for the analysis.

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Canonical Forms

$$\begin{bmatrix} a_1 & & 0 & & b_1 \\ & a_2 & & b_2 & \\ 0 & & \times & & 0 \\ & b_2 & & a_2 & \\ b_1 & & 0 & & a_1 \end{bmatrix}$$

Symmetric Persymmetric

$$\begin{bmatrix} 0 & & & & -a \\ & & & -b & \\ & & \diagdown & & \\ & b & & & \\ a & & & & 0 \end{bmatrix}$$

Skew-symmetric Persymmetric

$$\begin{bmatrix} a & & & & 0 \\ & b & & & \\ & & \ddots & & \\ & & & -b & \\ 0 & & & & -a \end{bmatrix}$$

Symmetric Perskew-symmetric

$$\begin{bmatrix} A_1 & & 0 & & B_1 \\ & A_2 & & B_2 & \\ 0 & & Z & & 0 \\ & -B_2 & & -A_2 & \\ -B_1 & & 0 & & -A_1 \end{bmatrix}$$

Skew-symmetric Perskew-symmetric