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Comparative dynamics in an overlapping-generations model: the effects of quasi-rational discrete choice on finding and maintaining Nash equilibrium

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Abstract Many models of Nash Equilibrium are complex enough that it becomes difficult to ascertain if and under what conditions the economic players can find and maintain this equilibrium. Using an analytical overlapping-generations model of goods, labor, and banking markets and quasi-rational discrete choice decision making, we find through agent-based simulations that Nash Equilibrium in goods market prices is at least locally stable when firms are sufficiently sensitive to changes in profits. In addition to verifying the analytical Nash outcome, the simulations verify that their economic agents, decision rules, and other protocols correspond to and maintain consistency with the analytical theory and identify important bounds of the analytical model.

Keywords Quasi rationality · Discrete choice · Life-cycle hypothesis · Nash equilibrium · Overlapping generations · Agent-based simulation

JEL classification C62 · C63 · D91

1 Introduction

The combination of rigorous mathematical theory and statistical empirics has provided great insight into many important issues surrounding Nash equilibria

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and economic theory as a whole. As these mathematical models become more complex, however, this approach sometimes has to focus more on the static properties of Nash and less the conditions under which an economic system can find or maintain it. As illustrated herein, agent-based simulation can be an important extension to the analysis of Nash equilibria, one that adopts established economic principles and yet provides opportunities for extension of Nash theory to new areas.

To give some background, conventional economic theory derives models from sets of accepted axioms regarding the preferences, objectives, and strategies of individual economic actors. These theories generate formal mathematical models in which axioms beget propositions, which are then supported by proofs; the models then generate closed-form equilibrium solutions whose robustness is analyzed with comparative statics. This approach is sound in its axioms, but can be self-limiting in its ability to explain a broader range of complex interactions that move real-world economic systems. For example, many real systems involve limited numbers of economic actors who make discrete choices in inherently probabilistic environments; under these conditions, the assumptions made in analytical models (generally to facilitate close-formed solutions) of infinitely divisible goods and twice-continuously differentiable objective functions may be too strong. Furthermore, while an analytical model may show that Nash equilibrium exists, it is not always clear whether its economic actors can find and maintain this equilibrium.

In comparison, agent-based economic modeling typically takes the form of sets of simulations in which the above axioms of preferences, objectives, and strategies of economic actors are modeled by independent computer programs (agents), which make economic decisions and interact in the economy over a sequence of time-steps. In general, this approach is consistent with analytical models, but agent-based modelers often fall short by not providing a theoretical blueprint of their economic model as a whole. Furthermore, such simulations may not constitute nor allow for the sort of general propositions and proofs provided by mathematical analysis, that is, they add little value without verification against formal theory. For the two approaches to add value to one another, we need a formal way to correspond them to one another.

A simulation is said to *correspond to* a formal theory if the variables and objectives of the agents correspond to those defined in the theory. Consider an analytical model consisting of firms seeking to maximize profit, a function of cost and revenue. A corresponding simulation could be composed of agents containing cost, revenue, and profit variables, as well as decision and interaction rules by which each agent seeks to maximize its profit variable. Decision rules can be as simple as an arithmetic function or as sophisticated as a genetic algorithm. There usually exist any number of decision rules and interaction rules that generate profit maximizing agents and therefore, in general, there exists a many-to-one correspondence of simulations to any given analytical model.

If a simulation that corresponds to an analytical model converges to the analytic solution, then the simulation is said to be *consistent* with the theory behind the analytical model. We call such a comparison a *consistency exercise*.[1] In a strict sense, consistency implies that there exists at least one set of parameterized conditions for which the analytical model and a corresponding simulation hold. It does not imply that all sets of parameters possible for a simulation will be consistent with the analytical model, and certainly does not imply that all simulations will be consistent.

We propose then an approach that begins with an analytical model and its analytically derived equilibrium solution, generates corresponding, and consistent simulations that test the assumptions and bounds of the analytical model, and compares the convergence of simulation calculations against the analytical solution. In a formal sense, we provide a verification of the mathematical model, analyze the model with comparative statics of the parameters *and* mechanisms by which Nash is achieved, and provide insight on what are the natural and important extensions to the analytical model.

The particular example is an overlapping-generations model of households, firms, a banking system and three markets: goods, labor, and money. To focus on whether the system can find and maintain a global optimum, Nash equilibrium is computed analytically in the goods market only (i.e., in this current version of the model, the other two markets play supporting roles), and then the agent simulations are used to explore variations in (1) the mechanisms by which firms select profit-maximizing prices and consumers find a low-price firm, (2) mechanisms in the supporting labor market, and (3) the overall parameters that drive the system.

1.1 Purpose and scope

The purpose of the article is to describe our approach of addressing a Nash equilibrium problem with an analytical model and then augmenting it with a simulation-based approach whose model corresponds to and is consistent with the analytical model. Section 2 describes the analytical version of the overlapping-generations model of firms, households, and banking system. It derives dynamic Nash equilibrium prices in the goods market, and then specifies parametric values in order to calculate an instance of this closed-form equilibrium solution. Section 3 describes an agent-based simulation composed of the same economic actors as in the analytical model and the same parametric values used to calculate Nash in the analytical model. In Sect. 4 we show that the results of the simulations are consistent with the analytically derived outcome and how these results are sensitive to changes in the mechanisms by which firms select selling price, how households search for low prices, and how firms

[1] Our notion of consistency exercises stems from the suggestion that simulation can serve as a means to theoretical discovery (see Ostrom 1988; Gilbert & Terna 2000; Luna & Stefansson 2000; McCain 2000, which are summarized in Hand, Paul, and Sprigg Jr 2005, Ch. 4). Additional rationales for the use of simulation relate to its potential for exploring models that are either extremely complex or have no closed-form solution, but these latter objectives are beyond the scope of this article.

adjust their labor. We then suggest some extensions to this analysis. Section 5 concludes with remarks and important extensions of this work.

2 Analytical model

Our analytical model builds upon the life-cycle economics work of Fisher (1930), Friedman (1957), Modigliani, and Brumberg (1955), and Ando and Modigliani (1963), and the overlapping-generations models of Samuelson (1958), Wallace (1980), Balasko, David, and Karl (1980), Balasko & Karl (1980, 1981a,b), and Tirole (1985) and McCandless and Wallace (1991). We model a discrete-time closed economy comprised of H households and F firms. Households decide how much to consume, borrow, and save each period. Firms decide whether to increase or decrease price and employment each period. Firms act as passive lenders in a banking market by making their cash reserves available for loans to households. There is no money creation. For the purpose of focusing our analysis on Nash in the goods market, we hold wages, productivity rates, interest rates, and marginal production costs as constant and equal across firms.

Households grow older with each time period, and face a lifespan comprised of an employment-eligible (career) phase during which households can earn wages in the labor market, and a retirement phase during which households can only consume by withdrawing funds from their private savings. Households cannot substitute intertemporally by accumulating goods, but they can borrow funds from firms or deposit savings with firms via a banking market. Banking allows households to smooth their consumption patterns over their lifespans according to a conventional life-cycle hypothesis.

Each firm seeks to maximize short-run profit by hiring labor from households via the labor market, and producing goods to sell to households in the goods market. Firms earn *nominal* profits by charging prices above marginal cost and by charging interest on loans. Firms earn *real* profits by spending nominal profits to purchase goods in the market and consuming them.

In this section, we derive general equations for the choice variables of households and firms. For households, we derive the optimal consumption expenditure and savings contribution for each period. For firms, we derive Nash equilibrium prices, where each firm's price is a function of its labor share and other firms' prices. We use these results in subsequent sections to calculate general equilibrium conditions. The analyses in this section use the constants in Table 1 and market variables in Table 2.

2.1 Household consumption and savings

We now derive the household's desired consumption expenditure and banking transaction in each time period. Consumption must be non-negative; a banking transaction can be either positive in the case of a deposit or negative in the case of a loan or withdraw. Table 3 lists the variables used to model the preferences, objectives, and strategies of the households.

Table 1 Constant parameters

ρ	$\equiv$ units of goods produced by unit labor per period
β	$\equiv$ consumption-elasticity of household utility
γ	$\equiv$ price-sensitivity exponent
λ	$\equiv$ periods per year
w	$\equiv$ wage rate
r	$\equiv$ market annual interest rate for bank loans and deposits
H	$\equiv$ number of households
F	$\equiv$ number of firms
Age_{min}	$\equiv$ minimum employment age
$\text{Age}_{\text{retire}}$	$\equiv$ mandatory retirement age
Age_{max}	$\equiv$ age of death

Table 2 Market variables

p_t	$\equiv$ price per unit goods; $p_t \geqslant 0 \in \{reals\}$
q_t	$\equiv$ units of goods; $q_t \geqslant 0 \in \{reals\}$
l_t	$\equiv$ units of labor; $l_t \geqslant 0 \in \{integers\}$
E_t	$\equiv$ number of employed households
L_t	$\equiv$ aggregate units of labor
Q_t	$\equiv$ aggregate units of goods
Y_t	$\equiv$ aggregate nominal wages (payrolls)
C_t	$\equiv$ aggregate household consumption expenditure
S_t	$\equiv$ aggregate household savings = deposits − debts
$\theta_{S,t}$	$\equiv$ aggregate household savings rate
M_t	$\equiv$ money

Table 3 Household variables

d_h	$\equiv$ household h's fixed discount rate for all periods t,
N_h	$\equiv$ periods retained in household h's memory of employment history,
$u_{h,t}$	$\equiv$ household h's utility,
$\phi_{f_0,t}$	$\equiv$ Pr[any household purchases goods for firm f_0],
$y_{h,t}$	$\equiv$ household h's income,
$c_{h,t}$	$\equiv$ household h's consumption expenditure,
$s_{h,t}$	$\equiv$ household h's increment savings $\equiv y_{h,t} - c_{h,t}$,
$b_{h,t}$	$\equiv$ household h's wealth $\equiv$ cash + deposits – debts, and
$\psi_{h,t>0}$	$\equiv$ Pr[the household is employed in future period t]

Each household is comprised at any given time of a single individual who becomes employment-eligible at Age_{min}, retires after $\text{Age}_{\text{retire}}$, and dies after Age_{max}. Let Age_0 denote a household's current age measured in years, where $\text{Age}_{\text{min}} \leq \text{Age}_0 \leq \text{Age}_{\text{max}}$. Time is discrete, with a fixed number of λ periods per year. We define T_0 as the number of periods for consumption before the household expires, where

$$T_0 \equiv \lambda(\text{Age}_{\text{max}} - \text{Age}_0) \left\{ \begin{matrix} > 1 & \forall \text{Age}_0 < \text{Age}_{\text{max}} \\ = 1 & \forall \text{Age}_0 = \text{Age}_{\text{max}} \end{matrix} \right\}. \quad (1)$$

We define K_0 as the number of time-steps for earning income before the household retires, where

$$K_0 \equiv \lambda(\mathrm{Age}_{\mathrm{retire}} - \mathrm{Age}_0) \left\{ \begin{array}{l} > 1 \ \ \forall \mathrm{Age}_0 < \mathrm{Age}_{\mathrm{retire}} \\ = 1 \ \ \forall \mathrm{Age}_0 = \mathrm{Age}_{\mathrm{retire}} \\ = 0 \ \ \forall \mathrm{Age}_0 > \mathrm{Age}_{\mathrm{retire}} \end{array} \right\}. \tag{2}$$

Any household that is employed by a firm supplies one unit of labor per period to its employer. All labor is supplied in discrete units, denoted as $l_t \in$ {positive integers}.

Each household derives utility in period t by consuming q_t units of goods. Utility is defined as $u_t = (q_t)^{\beta}$, where $\beta \equiv$ consumption elasticity $\in (0, 1)$, so that $u_t' > 0$ and $u_t'' < 0$. Each household valuates future consumption with respect to its internal discount rate, d_h, which implies that the current utility derived from expected future consumption is

$$u_0 = \frac{u_t}{(1+d)^t} = \frac{(q_t)^{\beta}}{(1+d)^t}, \tag{3}$$

such that

$$\frac{\partial u_0}{\partial q_t} = \beta \frac{(q_t)^{\beta-1}}{(1+d)^t}. \tag{4}$$

Households with a time-preference can increase utility by substituting consumption between time periods according to the following first-order condition, which must hold for any two time periods t_1 and t_2:

$$\frac{\partial u_0}{\partial q_{t_1}} = \frac{\partial u_0}{\partial q_{t_2}}. \tag{5}$$

Combining Eqs. 4 and 5 yields $\frac{q_t}{q_0} = (1+d)^{\frac{t}{\beta-1}}$, which provides the ratio of future-to-current consumption expenditure:

$$\frac{c_t}{c_0} = \frac{p_t q_t}{p_0 q_0} = \frac{p_t}{p_0} \cdot (1+d)^{\frac{t}{\beta-1}}. \tag{6}$$

2.1.1 Consumption and savings

Households optimize the present-value of current and future utility by setting their consumption and savings rates according to the conventional life-cycle hypothesis, represented by the following constrained-maximization problem:

$$\text{Maximize}\, u_{t=0} = \sum_{t=0}^{T_0} \frac{u_t\left(q_t\right)}{(1+d)^t}$$
$$s.t. \sum_{t=0}^{T_0} \frac{c_t^{\mathrm{e}}}{(1+r)^t} = b_{t=0} + \sum_{t=0}^{K_0} \frac{y_t^{\mathrm{e}}}{(1+r)^t}, \tag{7}$$

where b_0 denotes an initial wealth endowment, y_t^{e} is the expected nominal income earned in period t, and r denotes the market interest rate. We factor c_0 from the left side of the budget constraint from Eq. 7 to obtain

$$c_0 \cdot \sum_{t=0}^{T_0} \left(\frac{c_t^{\mathrm{e}}}{c_0} \cdot \frac{1}{(1+r)^t} \right) = b_0 + \sum_{t=0}^{K_0} \frac{y_t^{\mathrm{e}}}{(1+r)^t} \tag{8}$$

and substitute Eq. 6 into $\frac{c_t^{\mathrm{e}}}{c_0}$ to obtain

$$c_0 \cdot \sum_{t=0}^{T_0} \left(\frac{p_t^{\mathrm{e}}}{p_0} \cdot \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t} \right) = b_0 + \sum_{t=0}^{K_0} \frac{y_t^{\mathrm{e}}}{(1+r)^t}, \tag{9}$$

from which we obtain the optimal current consumption $\hat{q}_0$ and expenditure $\hat{c}_0$:

$$\hat{c}_0 = p_0 \hat{q}_0 = \frac{b_0 + \sum\limits_{t=0}^{K_0} \frac{y_t^{\mathrm{e}}}{(1+r)^t}}{\sum\limits_{t=0}^{T_0} \left(\frac{p_t^{\mathrm{e}}}{p_0} \cdot \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t} \right)}. \tag{10}$$

Since there is no money creation, we employ a simplifying assumption that expected prices are equal across time, which implies $\frac{p_t^{\mathrm{e}}}{p_0} = 1$. To summarize, in each period, each household will borrow or save to achieve current consumption of

$$\hat{c}_0 = p_0 \hat{q}_0 = \frac{b_0 + \sum\limits_{t=0}^{K_0} \frac{y_t^{\mathrm{e}}}{(1+r)^t}}{\sum\limits_{t=0}^{T_0} \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t}}. \tag{11}$$

The required savings transaction is

$$\hat{s}_0 = y_0 - \hat{c}_0, \tag{12}$$

where transactions are categorized as follows:

$$\begin{array}{lll} \textit{deposit}: & \hat{s}_0 \geq 0, & \forall b_0, \\ \textit{withdraw}: & \hat{s}_0 < 0 & \text{and } b_0 > 0, \\ \textit{loan}: & \hat{s}_0 < 0 & \text{and } b_0 \leq 0. \end{array} \tag{13}$$

For retired households, who cannot earn income, Eqs. 11 and 12 can, respectively, be simplified to

$$\hat{c}_0 = \frac{b_0}{\sum_{t=0}^{T_0} \frac{(1+d)^{\frac{t}{\beta-1}}}{(1+r)^t}} \quad \text{and} \quad \hat{s}_0 = -\hat{c}_0 \leq 0. \tag{14}$$

2.1.2 Selection of firm in the goods market

When more than one firm sells in the goods market, households must decide from which firm to purchase goods. In this model, each household randomly selects a firm each period, where firms with lower prices have higher probabilities of being selected. All households use the same firm selection rule, which is based on standard discrete-choice mechanics (McFadden 1974; Slepoy & Pryor 2002; Train 2003). Specifically, let ϕ be the selection probability:

$$\phi_{f_1} \equiv \text{Pr[household } h \text{ selects firm } f_1] \equiv \frac{p_{f_1}^{\gamma}}{\sum_{f=0}^{F} p_f^{\gamma}}, \tag{15}$$

where $\gamma < -1$ is a constant and p_f is the price charged by firm f. It can be shown that the relative probability of the household selecting firm f_1 over firm f_2 equals the scaled inverse of the firms' prices: $\frac{\phi_{f_1}}{\phi_{f_2}} = \left(\frac{p_{f_2}}{p_{f_1}}\right)^{|\gamma|}$.

2.1.3 Aggregate labor supply and goods demand

Since consumption of goods is the only source of utility for households, all employment-eligible households desire to work. Therefore, the aggregate household supply in the labor market is

$$L_t^{\mathrm{S}} = E_t = \sum_{h=1}^{H} l_h, l_h = \left\{ \begin{array}{l} 1 \ \forall h \ni \mathrm{Age}_0 \leq \mathrm{Age}_{\mathrm{retire}} \\ 0 \ \forall h \ni \mathrm{Age}_0 > \mathrm{Age}_{\mathrm{retire}} \end{array} \right\}. \tag{16}$$

Aggregate net demand for money is derived from Eq. 12:

$$M^{\mathrm{D}} = \sum_{h=1}^{H} \hat{s}_h(r, d_h, \beta_h, b_h, y, p). \tag{17}$$

Table 4 Firm variables

N_f	$\equiv$ periods retained in firm f's memory of profit history
$p_{f,t}$	$\equiv$ goods price offered by firm f
$l_{f,t}$	$\equiv$ units of labor employed at firm f
$\pi_{f,t}$	$\equiv$ nominal profit $\equiv$ revenue – cost
$\chi_{f,t}$	$\equiv$ real profit
$\sigma_{f,t}$	$\equiv$ firm f's labor share
$R_{f,t}$	$\equiv$ firm f's money reserves = cash + payroll + deposits − loans
$A_{f,t}$	$\equiv$ firm f's net loans to households = loans − deposits
$X_{f,t}$	$\equiv$ firm f's cash holding

Equation 11 implies that the aggregate household demand in the goods market is

$$Q^{\mathrm{D}} = \sum_{h=1}^{H} \hat{q}_h = \sum_{h=1}^{H} \frac{\hat{c}_{0,h}}{p_0}. \tag{18}$$

2.2 Firms and Nash Equilibrium pricing in the goods market

In order to focus on and isolate the effects of price changes in the goods market, we follow the New Keynesian assumption of sticky wages, where firms have limited ability to change the wage they offer to labor and thus can only adjust their quantity of labor. We explicitly assume that the wage rate w is constant across firms and across time. For analytical convenience, we also assume that productivity rate ρ is constant and equal across all firms. Firms search for their optimal labor quantity and product price that maximize current profits. Table 4 lists the variables used to model the preferences, objectives, and strategies of firms.

A firm uses l_t units of labor to produce and supply q_t^{S} units of goods to the goods market using the production technology

$$q_t^{\mathrm{S}} = \rho l_t. \tag{19}$$

Each firm sets its selling price p_t for goods, and earns *production* profit

$$\pi_t^{\text{production}} = \left(p_t q_t^{\text{sold}}\right) - w l_t \leq \left(p_t q_t^{\mathrm{S}}\right) - w l_t = (p_t \rho - w)\, l_t. \tag{20}$$

Firms also participate as passive lenders in a banking market by making their cash reserves available for loans to households at the fixed market interest rate r. Interest earned from loans equals rA_t and provides a second source of profit to the firm. Summarizing, a firm's total profit is defined by

$$\pi_t = \left(p_t q_t^{\text{sold}}\right) - w l_t + r A_t. \tag{21}$$

2.2.1 Employment and production

Each firm can be either supply constrained or demand-constrained. A firm is said to be supply constrained if $q_f^{\mathrm{S}} < q_f^{\mathrm{D}}$, since under this condition the amount that the firm can sell is limited by the amount of labor it can hire; that is $q_f^{\mathrm{sold}} = q_f^{\mathrm{S}} \equiv \rho l_f$. A firm is said to be demand-constrained if $q_f^{\mathrm{S}} > q_f^{\mathrm{D}}$, since under this condition the amount that the firm can sell is limited by demand for its goods; that is $q_f^{\mathrm{sold}} = q_f^{\mathrm{D}} \equiv \phi_f Q^{\mathrm{D}}$. Thus,

$$q_t^{\mathrm{sold}} = \min\left(\phi_f Q_t^{\mathrm{D}}, \rho l_t\right). \tag{22}$$

Equations 21 and 22 yield a profitability condition for price:

$$\pi_t > 0 \Leftrightarrow p_t \geq \max\left(\frac{wl_t - rA_t}{\phi_f Q^{\mathrm{D}}}, \frac{wl_t - rA_t}{\rho l_t}\right). \tag{23}$$

Supply constrained firms hire employees to maximize profit $\pi_t = (p_t\rho - w)l_t + rA_t$, where

$$A_t \equiv R_t - X_t - wl_t, \tag{24}$$

which implies

$$\pi_t = (p_t\rho - w)l_t + r\left(R_t - X_t - wl_t\right). \tag{25}$$

If the firm's reserves R_t exceed current wage payments and loans, then $X_t > 0$ and the marginal profit from additional labor is $\frac{\partial \pi_t}{\partial l_t} = (p_t\rho - w)$. However, if current wage payments and loans exhaust the firm's reserves, then $X_t = 0$ and the firm must forego interest from loans in order to hire labor. In this case, the marginal profit from additional labor is $\frac{\partial \pi_t}{\partial l_t} = (p_t\rho - (1+r)w)$. These results, in light of Eq. 19, provide the conditions under which profit-maximizing firms will try to obtain unlimited profits by hiring unlimited labor to produce unlimited goods:

$$\hat{q}_t = \rho \hat{l}_t = \left\{\begin{array}{lll} +\infty, & p_t > w/\rho & \text{and } X_t > 0 \\ 0, & p_t \leq w/\rho & \text{and } X_t > 0 \\ +\infty, & p_t > (1+r)w/\rho & \text{and } X_t = 0 \\ 0, & p_t \leq (1+r)w/\rho & \text{and } X_t = 0 \end{array}\right\}. \tag{26}$$

Once the labor pool is exhausted, supply constrained firms can only pursue profits via pricing, since by Eq. 25 profit is increasing with price: $\frac{\partial \pi_f}{\partial p_f} = \rho l_f > 0$. However, each firm's price is bounded by the market-share function in Eq. 15. Specifically, from

$$\left.\frac{\partial \phi_{f0}}{\partial p_{f0}}\right|_{\bar{p}_{f\neq f0}} = \frac{\gamma\phi(1-\phi)}{p_{f0}} < 0. \tag{27}$$

It follows that increases in a particular firm's price will reduce ϕ_f until $\rho l_t \geq \phi_f Q^{\mathrm{D}}$ at which time the firm becomes demand-constrained rather than supply constrained. By Eqs. 22 and 23, the demand-constrained firm has profit

$$\pi_t = p_t\phi_t Q^{\mathrm{D}} - wl_t + rA_t. \tag{28}$$

This equation is used later to derive the Nash equilibrium price distribution in the goods market.

2.2.2 Aggregate goods supply and labor demand

Equation 19 implies that the aggregate supply of goods is

$$Q_t^{\mathrm{S}} = \sum_{f=1}^{F} \rho l_{f,t} = \rho E_t. \tag{29}$$

Equations 22 and 26 imply that firms' aggregate demand for labor is

$$L^{\mathrm{D}} = \sum_{f=1}^{F} \hat{l}_f = \sum_{f=1}^{F} \frac{\hat{q}_f}{\rho} = \{0, +\infty\}. \tag{30}$$

We derive the firms' aggregate supply of money from Table 4 and Eq. 24:

$$M^{\mathrm{S}} = \sum_{f=1}^{F} [R_f - wl_f]. \tag{31}$$

2.3 Market equilibria

The necessary conditions for market clearing are

$$Q^{\mathrm{S}} = Q^{\mathrm{D}}, \quad L^{\mathrm{S}} = L^{\mathrm{D}} \quad \text{and} \quad M^{\mathrm{S}} = M^{\mathrm{D}}. \tag{32}$$

2.3.1 Nash equilibrium prices in the goods market

Under the discrete choice defined in Eq. 15, equilibrium prices will vary across firms as the scaled inverse of labor shares. To show this, we first note from Eq. 12 that $\sum_{h=1}^{H} \hat{c}_{h,t} = \sum_{h=1}^{H} y_{h,t} - \sum_{h=1}^{H} \hat{s}_{h,t}$, which implies

$Y_t = S_t + \sum_{h=1}^{H} \hat{c}_{h,t} = \theta_{S,t} Y_t + \sum_{h=1}^{H} \hat{c}_{h,t}$, where $\theta_{S,t} \equiv \frac{S_t}{Y_t}$. Since $Y_t = wL_t$, we have $\sum_{h=1}^{H} \hat{c}_{h,t} = (1 - \theta_{S,t})wL_t$. Substituting into Eq. 18 yields

$$Q_t^{\mathrm{D}} = \frac{(1 - \theta_{S,t})wL_t}{p_t}. \tag{33}$$

By definition, the market-clearing condition for an individual firm f is $q_f^{\mathrm{D}} = q_f^{\mathrm{S}}$. By Eqs. 19 and 22, we have

$$q_f^{\mathrm{D}} = q_f^{\mathrm{S}} \Leftrightarrow \phi_f Q^{\mathrm{D}} = \rho l_f \Leftrightarrow \phi_f = \frac{\rho l_f}{Q^{\mathrm{D}}}. \tag{34}$$

Substituting Eq. 33 into 34 yields $\phi_f = \frac{\rho l_t p_t}{(1-\theta_{S,t})wL_t}$. Incorporating Eq. 15 and Table 4 provides the Nash equilibrium price, given by

$$q_f^{\mathrm{D}} = q_f^{\mathrm{S}} \Leftrightarrow p_f^* = \left[\frac{\rho \cdot g_t \cdot k_t \cdot \sigma_{f,t}}{w}\right]^{\frac{1}{\gamma-1}}, \tag{35}$$

where $k_t = \sum_{f=1}^{F} p_{f,t}^{\gamma}$ and $g_t = \frac{1}{1-\theta_t}$. That is, given its labor share $\sigma_{f,t}$ and the other firms' prices k_t, firm f cannot benefit by charging any price different from p_f^*. The following partial derivative shows that equilibrium prices vary as the scaled inverse of labor share:

$$q_f^{\mathrm{D}} = q_f^{\mathrm{S}} \Leftrightarrow \frac{\partial p_f^*}{\partial \sigma_f} = \left(\frac{1}{\gamma-1}\right)\left[\frac{\rho \cdot g_t \cdot k_t}{w}\right]^{\frac{1}{\gamma-1}} \sigma^{\frac{\gamma}{1-\gamma}} < 0. \tag{36}$$

Equations 35 and 36 imply that smaller firms are able to settle at higher prices; the reason is that smaller firms produce fewer goods and therefore require a smaller market share to maximize profit.

2.3.2 Banking equilibrium

The banking industry clears with no tradeoffs between wages and loans when money supplied by firms covers the money demanded by households. Equations 17 and 31 imply the clearing condition,

$$M_t^{\mathrm{D}} \le M_t^{\mathrm{S}} \Leftrightarrow \sum_{h=1}^{H} \hat{s}_{h,t} \le \sum_{f=1}^{F} \left[R_{f,t} - wl_{f,t}\right], \tag{37}$$

which by Table 2 implies

$$M^{\mathrm{D}} \le M^{\mathrm{S}} \Leftrightarrow S_t + Y_t \le \sum_{f=1}^{F} \left[R_f\right]. \tag{38}$$

Therefore, the banking market clears with no wage-loan tradeoffs as long as firms' total reserves exceed households' incomes and net savings. Note that Y_t is non-negative, but S_t is negative if households are net borrowers.

3 Agent-based simulation

The prescribed model links the static equilibrium in the goods market to the dynamic equilibrium in the financial market, and derives the Nash equilibrium in the goods market as a function of consumers' price sensitivity, represented, by γ. A higher value for γ implies a household has a higher probability of selecting the firm with the lowest price (see Eq. 15). If we were to assume that search costs are zero, then γ might be interpreted as the degree of rationality of quasi-rational consumers. This section describes a computer simulation of the analytical model in which the economic actors use a set of preferences, objectives, and strategies that correspond to the analytical model, but differ in that firms in the simulation use quasi-rational adaptive search to search for their profit-maximizing behavior. Specifically, we define a *rational* firm as one with unbounded profit sensitivity (that is, a firm that *always* selects the pricing strategy associated with highest profit) and a *quasi-rational* firm as one with bounded profit sensitivity (that is, a firm that *probabilistically* selects the pricing strategy associated with highest profit). Profit sensitivity is shown to be a pivotal factor in determining whether firms can collectively find and maintain the Nash equilibrium.

3.1 Mechanics

We implemented this simulation in a modified version of the Aspen agent-based modeling software developed at Sandia National Laboratories; see Basu, Pryor and Quint (1998) for details on the structure and uses of Aspen. The simulation uses independent computer programs to simulate households, firms, and a bank; these computer programs interact by passing messages in a discrete-time environment.

All agents maintain private information and interact via message passing. At the start of each time period, each agent conducts an assessment of its state variables and formulates a list of objectives and action items. Action items generally involve transactions with other agents. Some transactions require iterative communication. For example, a household seeking employment must send a job application to a firm, which responds with an offer or rejection. Subsequently, the household must accept or reject any job offers. To allow for proper assessment and interaction, each time period is divided into an assessment step in which all agents set their objectives and action items, followed by several messaging steps to allow agents to complete transactions during the time period.

When agents must make a discrete choice, each option has a probability of selection. We use pseudo-random numbers to stochastically select from a set of options. Numbers are drawn sequentially and staggered across agents as the simulation cycles through each time period. Pseudo-random numbers are sufficient for this exercise, in which we seek to observe whether or not the prescribed discrete-choice mechanics will cause/allow the simulation to perpetually converge to Nash equilibrium.

3.2 Agents

The economic actors in the model are represented by autonomous adaptive agents who store private information and make economic decisions. Firms and household actors make decisions in pursuit of clearly defined economic objectives. Each household repeatedly examines its own employment history to modify expectations for future employment, which are used to make consumption decisions based on discounted cash-flow criteria. Each firm compares its recent profit levels with its profit history, and uses decision rules to perpetually modify its pricing strategy.

3.2.1 Bank

A single bank agent serves as financial intermediary between households and firms. The bank holds the firms' excess reserves, which are made available as loans to households. The bank establishes an account for each household. Account balances can be positive or negative. Positive balances represent deposits, and accrue interest for the household at market interest rate r. Negative balances represent loans, and accrue interest for the bank at market interest rate r.

For simplicity, we assume that the bank applies the same interest rate to both loans and deposits, which implies that the bank, and therefore firms, should only engage in banking if households are net borrowers in the aggregate. A more general implementation of banking would allow for a spread between lending and saving rates, but is beyond the scope and purpose of this exercise. We ensure integrity in the banking portion of the simulation by specifying a distribution of households whose aggregate savings profile is negative, and by setting initial reserves to satisfy Eq. 38.

3.2.2 Households

Households maintain the variables defined in Table 3. At the start of each time period, each household makes two primary assessments. First, if an employment-eligible (career) household is unemployed, then it sends a job application to a firm. Second, all households calculate their target consumption and savings according to Eqs. 11 and 12. Equation 12 is calculated in part using income in the current period, which is known:

$$y_{h,t=0} = \left\{ \begin{array}{ll} 0, & \text{if unemployed} \\ w, & \text{if employed} \end{array} \right\}. \tag{39}$$

However, the calculation also requires career households to make assumptions regarding future income. In this simulation, future income is derived from employment history as follows. Each household keeps a record of its employment history, and calculates the average number of periods employed to estimate the probability of being employed in any future time period:

$$\Pr[\text{employed in future period } t] \equiv \sum_{t=1}^{N_h} \frac{y_{-t}}{w}, \tag{40}$$

where N_h denotes the number of periods retained in the household's memory. The household then probabilistically projects its income in each future period based on Eq. 40 as follows:

$$y^{\mathrm{e}}_{t>0} = \left\{ \begin{array}{l} w, \ \text{ with } \Pr[\text{employed in } t] \\ 0, \ \text{with } \Pr[\textit{un}\text{employed in } t] \end{array} \right\}. \tag{41}$$

When a household retires, it retains its employment history, which is provided to its descendent. Thus, each new entrant initially projects its future employment based on its parent's history, then increasingly updates its history with its own experience.

The household uses employment projections to calculate its consumption expenditure according to Eq. 11.[2] Once the consumption expenditure is determined, the household selects a firm from which to purchase goods according to Eq. 15, sends a purchase order to the selected firm, and wires a bank transaction in accordance with Eq. 12 for a deposit, withdraw, or loan.

We instantiate households with a uniform age distribution ranging from age 20 to 80. Each household is comprised of a single individual who becomes employment-eligible at age 20, retires after age 60, and dies after age 80. There are $\lambda = 5$ time periods per year, resulting in 301 periods per each individual's lifespan. The simulation instantiates 301 households with a uniform age distribution, so there is exactly one household associated with each period in the life cycle. Each household has one descendent who becomes employment-eligible when its parent dies, and which inherits any remaining debts or bank deposits. Under this framework, the age distribution is cyclical and corresponds to the initial age distribution. Additionally, all households have the same discount rate and consumption elasticity. The combined assumptions of uniform age distribution and equal discount rate and consumption elasticity greatly simplify calculations of the expected aggregate consumption/savings profile: that is, summing the periods of the discounted cash-flow profile for a single household

[2] Note that $y^{\mathrm{e}}_{t>0}$ is a stochastic, and is not equivalent to the expected value: $EV[y_{t>o}] = w \cdot \Pr[\text{employed in } t]$. Stochastic expectations introduce a degree of variance into the households' consumption choices to better test the robustness of convergence to Nash equilibrium.

provides the expected aggregate profile for the population of households. Since all career households desire jobs, the expected aggregate employment level, E, equals the number of career households. Since households do not expect price changes, their consumption expenditures and savings levels are independent of goods prices.

The mechanics of this model allow households to hold money as either cash or bank deposits. Bank deposits earn interest at the market rate. Wages are paid in cash and do not earn interest in the period they are received by the household, but may be deposited to earn interest in the future.

3.2.3 Firms

Firms maintain the variables defined in Table 4. To explore the relationship described in Eq. 35, we ensure cross-sectional variation in the labor share by assigning each firm an initial target number of employees derived from the share function: $f/\sum_{f=1}^{F} f$.

At the start of each time period, each firm decides whether to increase or decrease its price for goods. Firms must perpetually search for optimal price using a simple algorithm. Firms make their pricing decision by assessing a running record of profits[3] for the previous N_f periods, and altering their strategies for *scaling* their prices. At the start of each period, each firm calculates *recent* profits $\sum_{t=1}^{N_f/2} \pi_{-t}$ and *bygone* profits $\sum_{t=(N_f/2)+1}^{N_f} \pi_{-t}$. The firm also knows its scaling strategy from the previous period, represented by a scaling factor $\delta_{-1} = \frac{p^*_{-1}}{p^*_{-2}} \in (0, 2]$, where $\delta_t \in (0, 1)$ denotes a price decrease in period t, and $\delta_t \in (1, 2]$ denotes a price increase.

The firm chooses in each period whether to (1) *re-adopt* its strategy by applying the scaling factor used in the previous period: $p^*_0 \equiv \delta_0 \cdot p^*_{-1}$, where $\delta_0 \equiv \delta_{-1}$, or (2) *reverse* its strategy by reverting to the price associated with the highest profits in memory: $p^*_0 \equiv p^*_t \sim \max\{\pi_t\}_{t=-1}^{-N_f}$. In this latter case, the firm notes its reversal by resetting its pricing strategy: $\delta_0 = \frac{p^*_0}{p^*_{-1}}$. The firm stochastically decides whether to re-adopt or reverse based on discrete-choice mechanics, where the probability of each choice increases with the profits associated with that choice, as given by

$$\Pr[\text{re-adopt strategy}] \equiv \frac{\pi^{\nu}_{\text{new}}}{\sum\limits_{i=\{\text{new},\text{bygone}\}} \pi^{\nu}_i} \quad \text{and}$$
$$\Pr[\text{reverse strategy}] \equiv 1 - \Pr[\text{re-adopt strategy}], \tag{42}$$

[3] For purposes of generality, firms in this simulation search for the price that maximizes *real* profits, defined as $\chi \equiv \pi/p$, which remains consistent with the conditions of Eq. 26.

where $\nu \geq 1$ determines firms' sensitivity to profit. A higher value for ν implies a firm has a higher probability of selecting the strategy associated with higher profits. In this simulation, the value of ν can be interpreted as the degree of rationality of quasi-rational firms.

In addition to an adaptive pricing strategy, the pricing algorithm also allows for two corrective adjustments. First, if a firm did not change its price in the previous period, then the firm tests its current strategy by randomly either increasing or decreasing its price by one-half percent. Second, if $p_{-1} > 0$ and $q_{-1}^{\text{sold}} = 0$, then the firm reduces price by one percent.

Each firm searches for the optimal production scale (i.e., quantity of labor) in the same way that it searches for optimal price by repeatedly assessing a running record of profits[4] for the previous N_f periods and altering its strategy for *scaling* its labor force. At the start of each period, each firm calculates *recent* profits $\sum_{t=1}^{N_f/2} \pi_{-t}$ and *bygone* profits $\sum_{t=(N_f/2)+1}^{N_f} \pi_{-t}$. The firm also knows its scaling strategy from the previous period, represented by a scaling variable $\delta_{-1} = \frac{l^*_{-1}}{l^*_{-2}} \in (0, 2]$, where $\delta_t \in (0, 1)$ denotes "layoffs" in period t, and $\delta_{-1} \in (1, 2]$ denotes attempts to hire new employees in period t. If recent profit are greater than bygone profits, then the firm re-adopts its strategy by applying the scaling factor used in the previous period: $l^*_0 \equiv \delta_0 \cdot l^*_{-1}$, where $\delta_0 \equiv \delta_{-1}$. If recent profits are lower than bygone profits, then the firm reverts to the price associated with the highest profits in memory: $l^*_0 \equiv l^*_t \sim \max\{\pi_t\}_{t=-1}^{-N_f}$. In this latter case, the firm notes its reversal by resetting its labor strategy: $\delta_0 = \frac{l^*_0}{l^*_{-1}}$.

In addition to an adaptive labor strategy, the labor algorithm also allows for two corrective adjustments. First, if the firm did not change its labor force in the previous period, then the firm tests its current labor strategy by randomly either increasing or decreasing its target labor force by one employee. Second, if a firm's target labor force falls to zero employees, then the firm resets its target labor force to one employee.

4 Simulation results

We conducted three different sets of simulations, each of which specifies different parameters and agent mechanics under which to observe whether the simulation finds and maintains Nash equilibrium in the goods market, and additionally the households' optimal savings rates in the financial market. The first set of simulations is entirely deterministic except for the aforementioned firm selection rule. The second set introduces quasi-rationality into the firm's adaptive pricing algorithm. The third set introduces adaptive labor scaling allowing each firm to adjust its quantity of labor over time. Table 5 lists the input parameters used in these simulations.

[4] For purposes of generality, firms in this simulation search for the labor-scale that maximizes *real* profits, defined as $\chi \equiv \pi/p$, which remains consistent with the conditions of Eq. 26.

Table 5 Simulation parameters

	Parameters	Symbol	Value
Global			
	Number of time periods	-	2,000
	Number periods per year	λ	5
	Wage rate	w	50
	Market interest rate	r	5.0%
Households			
	Number of households	H	301
	Consumption-elasticity of utility	β	0.3
	Price-sensitivity exponent	γ	$\{-2, -3, -4, -5, -9\}$
	Discount rate	d	4.0%
	Minimum employment age	Age_{min}	20
	Mandatory retirement age	$\text{Age}_{\text{retire}}$	60
	Expiration age	Age_{max}	80
	Age distribution (uniform)	$\{\text{Age}_{h,t=0}\}H$	$\sim$[20,80]
	Employment-eligible households	E	201
	Length of memory (# periods)	N_h	$3 \cdot \lambda$
Firms			
	Number of firms	F	5
	Productivity rate	ρ	2
	Profit-sensitivity exponent	ν	See Table 7
	Initial labor share	$\sigma_{f,t=0}$	$f/\sum_{f=1}^{F} f$
	Initial reserves	$R_{f,t=0}$	\$200K
	Length of memory (# periods)	N_f	$2 \cdot \lambda$

Since the analytical model explicitly calls for households to use the quasi-rational firm selection rule defined by Eq. 15, all simulations include this feature as part of the households' decision process.

4.1 Set #1: rational adaptive pricing with fixed labor

The baseline set of simulations is entirely deterministic, except for the quasi-rational firm selection rule. Firms employ rational adaptive pricing ($\nu = \infty$) and therefore deterministically (always) select the pricing strategy associated with the highest profits. Firms are also restricted from laying off employees or from altering their initial target number of employees derived from the share function: $f/\sum_{f=1}^{F} f$. Thus, workers never experience layoffs and households' expectations are always formed by $y_t^{\mathrm{e}} = w$ in Eq. 41. These baseline conditions provide both real and perceived job security for households, allowing them to optimize their time-dependent lifetime consumption.

Based on the assumption of job security and the households' input parameters in Table 5, the aggregate optimal consumption and savings were calculated in a spreadsheet and listed in Table 6. This table also lists the average

Table 6 Calculated household optima

Case	Consumption expenditure			Bank balances		
	Career	Retired	All	Career	Retired	All
Optima	\$35.89	\$4.38	\$25.42	−\$1235	\$133	−\$780
Fixed labor						
Baseline	\$35.88	\$4.38	\$25.41	−\$1235	\$133	−\$780
Adaptive pricing	\$35.88	\$4.38	\$25.41	−\$1235	\$133	−\$780
Adaptive labor						
"Rational"	\$35.70	\$4.52	\$25.34	−\$1208	\$138	−\$761
Optimistic	\$34.90	\$4.28	\$24.73	−\$1281	\$130	−\$812

consumption and savings for the baseline simulation, which clearly converge to the expected optima.

By Eq. 35, we do not expect a single market equilibrium price, but rather a range of prices that vary across firms with respect to labor share as a function of γ. To demonstrate the robustness of convergence, we will run five baseline simulations assuming five different price-sensitivity exponents: $\gamma = -2, -3, -4, -5$, and -9. Figure 1 shows price formations for three of the five assumed values of γ. The left column shows the formation of each firm's price over time. The right column shows a scatter plot of each firm's prices and labor shares. Each scatter plot also includes a curvilinear line-of-fit showing the log-linear relationship between prices and labor share. These plots demonstrate that (1) smaller firms converge to higher prices, (2) higher consumer price sensitivity results in lower average price and lower cross-firm deviation in prices, and (3) prices form a log-linear relationship with labor share. Consistent with theory, these outcomes show that increasing consumer price sensitivity reduces firms' ability to assert market power, and thereby forces Nash equilibrium prices down closer to the competitive equilibrium set of prices.

We now conduct a more rigorous examination of price formation to observe the perpetual regression of prices to Nash equilibrium. We first rewrite Eq. 35 in natural logarithms to obtain the linear representation

$$\ln(p_{f,t}) = \eta \ln(\rho/w) + \eta \ln(g_t) + \eta \ln(k_t) + \eta \ln(\sigma_{f,t}), \tag{43}$$

where $\eta \equiv \frac{1}{\gamma-1}$. Noting that $\frac{\rho}{w}$ is constant and g_t and k_t are equal for all firms in any period t, we can rewrite Eq. 43 as a point-in-time relationship between price and labor share:

$$\ln(p_{f,t}) = \alpha_t + \eta_t \ln(\sigma_{f,t}). \tag{44}$$

Using the regression command in Stata (see StataCorp 2003), we obtain the least-squares fitted value of $\hat{\eta}_t$ for each period. From these estimates, we calculate the estimated price-sensitivity coefficient in each period: $\hat{\gamma}_t = \frac{1+\hat{\eta}_t}{\hat{\eta}_t}$. Since the true household price-sensitivity coefficient γ is known, a favorable

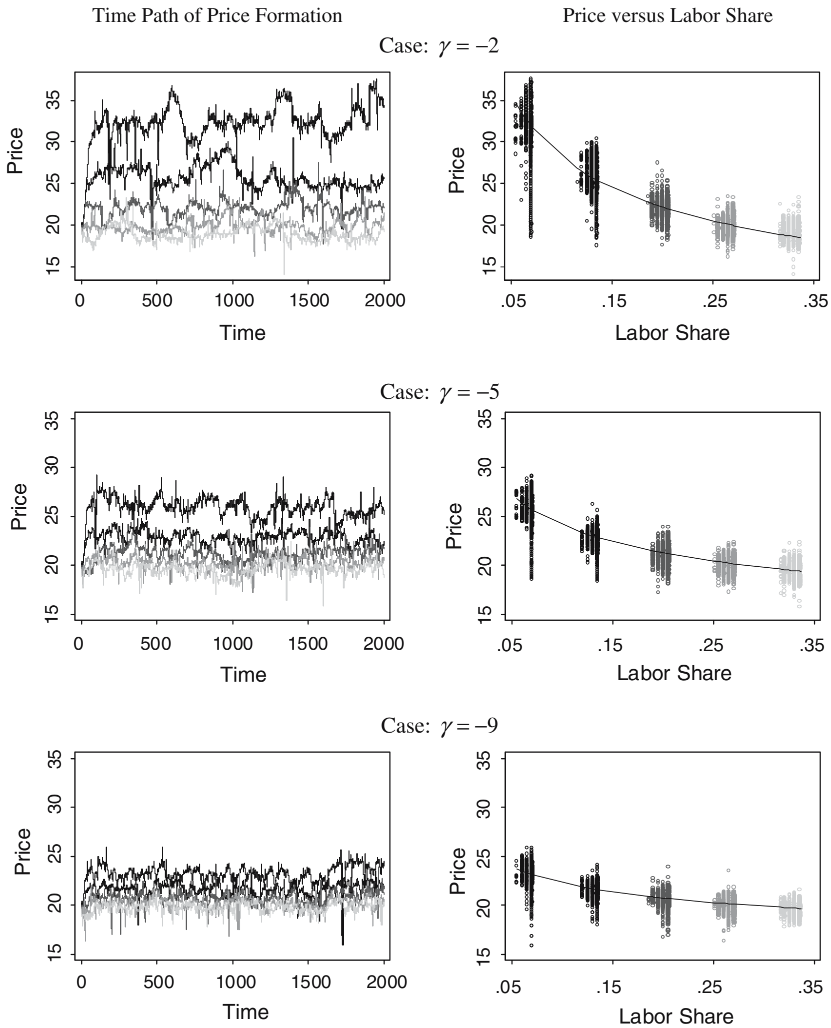

Fig. 1 Nash Equilibrium price formation for five competing firms with different labor shares (distinguished by *shade*)

comparison of $\hat{\gamma}_t$ and γ will imply convergence to Nash equilibrium. Figure 2 plots the fitted $\hat{\gamma}_t$ in all periods for three simulations in which $\gamma = -2, -5$, and -9. The plots clearly show for each simulation that $\hat{\gamma}$ perpetually fluctuates about γ over time, suggesting that prices perpetually regress toward Nash equilibrium.

4.2 Set #2: quasi-rational adaptive pricing

The second set of simulations relaxes the baseline assumption of $\nu = \infty$ and instead simulates price formation under bounded profit-sensitivity exponents.

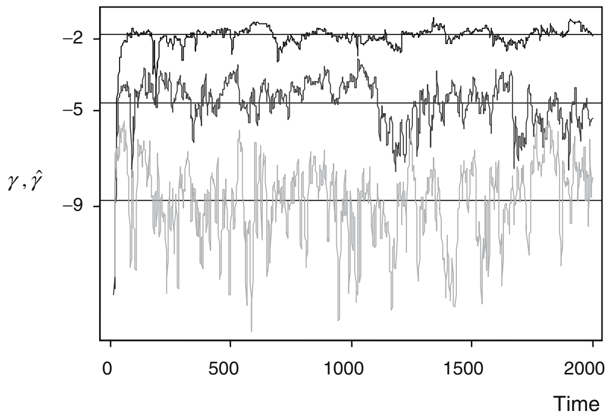

Fig. 2 Fitted (observed) $\hat{\gamma}$ from three simulations corresponding to three different input values for γ

Comparisons of rational and quasi-rational adaptive pricing will assess whether firms' profit sensitivity affects their ability to assert market power. That is, just as consumers' price sensitivity forces prices toward the competitive equilibrium, so we expect firms' profit sensitivity to force prices away from the competitive equilibrium. If $\hat{\gamma}_\nu \rightarrow \gamma$ for all ν, then we might reasonably conclude that ν does not affect the firm's ability to assert market power. To explore this relationship, we ran the simulation several times to identify for which, if any, values of $\nu < \infty$ the simulation fails to converge to Nash. We compare the baseline outcome to quasi-rational outcomes under ν =50, 75, and 150.

The only difference between the baseline simulations and these quasi-rational adaptive pricing simulations is the profit sensitivity exponent, which affects the probability of choosing the "correct" strategy. To properly compare the outcomes under baseline and quasi-rational adaptive pricing, we systematically seed the pseudo-random number generator in each period in such a way to ensure that a particular firm always draws the same number for a particular decision in a particular period. Of course, this feature does not ensure the firm will make the same choice, because the likelihood of a "correct" depends on profit sensitivity. Indeed, that is the point of introducing quasi-rational adaptive pricing. However, this feature ensures that different outcomes arise strictly due to the cumulative effect of profit sensitivity, and not due to differences in the pseudo-random number sequence.

Table 7 compares the average fitted $\hat{\gamma}$ for several profit sensitivity exponents. We find that $\hat{\gamma}$ usually converges close to γ for $\nu \geq 75$, but that outcomes usually diverge away from Nash equilibrium at $\nu = 50$. This exercise demonstrates that convergence to Nash equilibrium depends on the firms' profit sensitivity.

In this model, any divergence from Nash equilibrium affects the quantity of units consumed by households, but not their financial decisions. The average household consumption expenditures and bank balances conform to the same optima as in the baseline model, as shown in Table 6.

Table 7 Price formation under quasi-rational adaptive pricing.

Profit sensitivity		γ				
		−2.00	−3.00	−4.00	−5.00	−9.00
$\nu = \infty$		−1.81	−2.89	−4.04	−4.56	−8.74
$\nu = 150$		−1.75	−2.60	−3.75	−4.68	−8.86
$\nu = 75$	$\hat{\gamma}$	−1.72	−2.50	−1.59	−4.36	−6.71
$\nu = 50$		−0.34	−1.58	−1.33	−1.01	−7.24

4.3 Set #3: adaptive labor scaling

The third set of simulations relaxes the baseline assumption that labor is fixed. Here, firms apply the adaptive labor scaling algorithm described in Sect. 3.2.3. This algorithm allows for layoffs, which subsequently affect households' expectations and consumption profiles, and importantly, how they purchase in the goods market.

4.3.1 Effects of labor scaling on pricing

Labor adjustments cause labor shares to change over time. Figure 3 compares price formation under labor adjustments with the baseline plots copied from Fig. 1. These plots show that firms' labor shares and size rankings change substantially under labor scaling. However, the plots retain the baseline characteristics that (1) smaller firms achieve higher prices, (2) price deviation declines as price sensitivity increases, and (3) prices form a log-linear relationship with labor share.

Table 8 lists the fitted $\hat{\gamma}$ for values of the sensitivity parameters. As with fixed labor, we find that prices converge to Nash equilibrium at a sufficiently high level of profit sensitivity.

4.3.2 Unemployment and consumer confidence

The adaptive labor search creates layoffs and an expectation of future layoffs, which causes currently or recently unemployed households to adjust consumption and savings accordingly. We can observe these adjustments in this simulation by comparing the households' financial profiles with those in the baseline simulation, in which households enjoyed both real and perceived job security.

Figure 4 shows the average household financial profile with respect to age assuming $\gamma = -2$ and $\nu = \infty$. In this exercise, all households have a fixed positive discount rate, resulting in greater planned consumption in earlier years. The upper plot shows the household's optimal planned consumption expenditure. Younger households achieve the optimal consumption by borrowing against future earnings. The lower plot shows the corresponding planned bank transactions required to achieve the consumption schedule shown in the upper

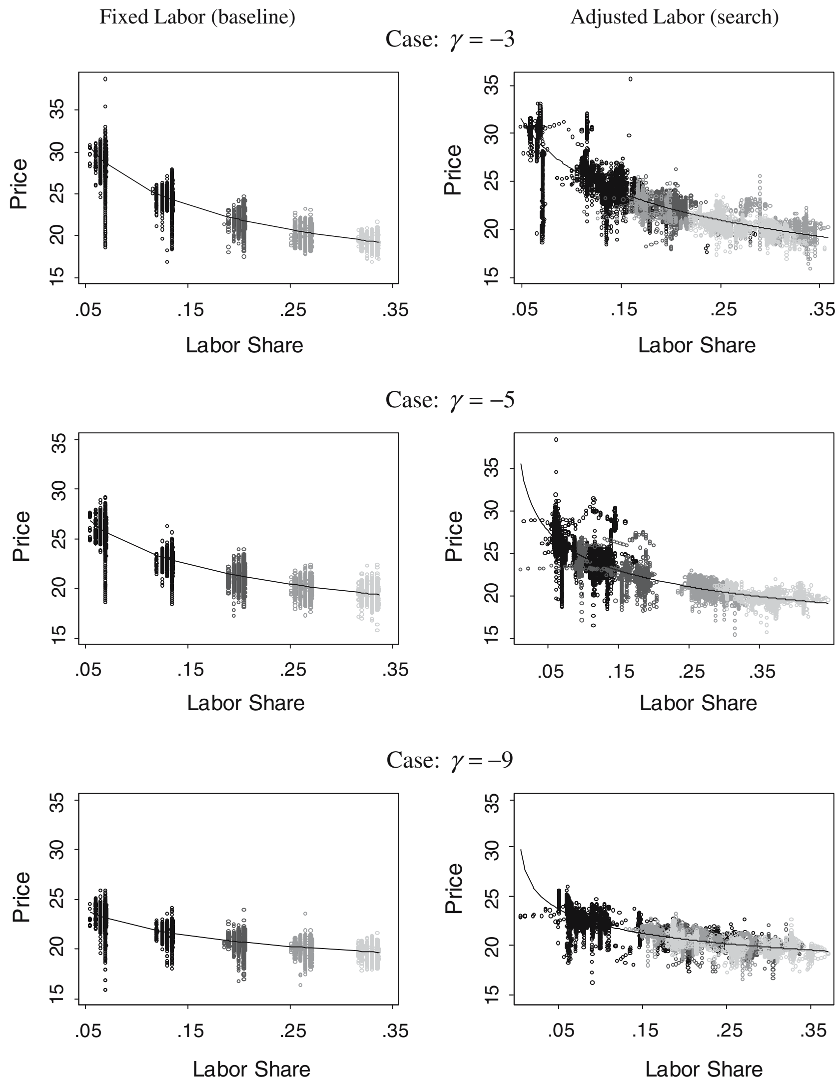

Fig. 3 Price formation with fixed and adjusted labor for five competing firms with different initial labor shares (distinguished by *shade*)

plot. We see that each household will borrow loans through age 38, make loan payments and deposits from age 38 to 60, and make withdraws after retirement at age 60. These plots show that adaptive labor search causes wary households restrict consumption in the earliest stages of the life cycle (upper plot) to reduce their debt stream (lower plot). Table 6 (see *Adaptive Labor: "rational"*) confirms that households consume and save less when layoffs are allowed.

In this simulation, households form "rational" expectations of a sort by incorporating their employment history into their expectation for future in-

Table 8 Price formation under adaptive labor search

Profit sensitivity		γ				
		−2.00	−3.00	−4.00	−5.00	−9.00
$\nu = \infty$		−1.79	−2.72	−3.81	−4.71	−9.13
$\nu = 150$		−1.86	−2.23	−3.65	−4.44	−7.59
$\nu = 75$	$\hat{\gamma}$	−1.71	−0.09	−3.73	−4.90	−35.38
$\nu = 50$		–	–	–	–	–

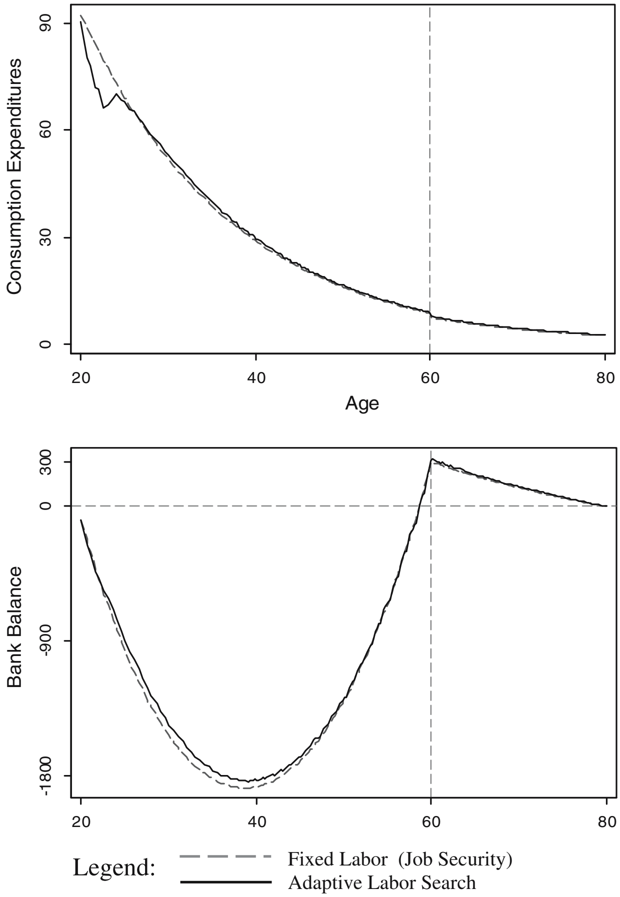

Fig. 4 Unemployment and consumer confidence

come. For further comparison, we executed an additional simulation in which households were fully optimistic concerning future employment regardless of their consumption history. Table 6 shows that optimistic households borrow more resulting in lower consumption during both their career and their retirement.

This comparison demonstrates that firms' search for optimal production scale creates non-cyclical unemployment, which affects the households' financial profiles. The nature and magnitude of those effects, however, depend on households' memory of and attitude toward unemployment.

5 Summary and future work

This exercise demonstrates the robustness of theory in a complex system of interrelated agents and markets. Specifically, we find that firms with limited memory, no public information, and very simplistic decision processes can "discover" their Nash equilibrium prices despite various sources of noise and uncertainty. Table 9 emphasizes the importance of convergence by highlighting the information that is unavailable to the firms in the course of their discovery. The firms converge despite relative ignorance and a reliance on extremely rudimentary search algorithms. Thus, the simulation supports the robustness and validity of the analytical model.

This exercise also identifies firms' profit sensitivity as a pivotal factor for convergence to Nash equilibrium; when profit sensitivity is too low, the goods market fails to converge to Nash equilibrium.

As a means of assessing the value that the agent-based simulation adds to the analytical model, we introduce some criteria listed in Table 10. Under these criteria, this exercise achieves moderate success: it identifies a range of conditions under which the simulation converges to an analytically derived solution, but also identifies conditions where it fails to converge. These findings suggest a need for an extension to the analytical model. Specifically, firms' profit sensitivity should be added as an endogenous component within the model. However, profit sensitivity, as a factor for finding Nash equilibrium, also depends on the length of the firm's memory, and the run-time inter-temporal variance in firm profits. These complex relationships are difficult to capture in an analytical framework, whereas they were rather easily incorporated into the simulation to discover the tipping point for profit sensitivity at which Nash breaks down.

The existing simulation leaves several questions for future consideration. First, the length and weighting of memory affects both households' expectations and firms' pricing and labor decisions. So, the role and interplay between memory and profit sensitivity warrant deeper investigation. Second, Fig. 2 indicates

Table 9 Information unavailable to firms

Number of other firms
Prices charged by other firms
Size of other firms
Number of consumers (i.e., market share)
Number of laborers (i.e., labor share)
Selection rule used by households; see Eq. 15
Consumer price sensitivity
Consumer churn/retention rates
Nash equilibrium conditions

Table 10 Exercise assessment criteria

Success rating	Criteria
Minimum	The simulation converges to the analytically derived solution, and allows for meaningful (non-trivial) observations of the agent characteristics and environmental conditions that allow for convergence
Moderate	Variations of a simulation are run, some of which converge to an existing closed-form solution and some of which do not. Pivotal parameters, rules, or interaction criteria are identified, allowing for postulates and subsequent investigation
High	Variations of a simulation are run, some of which converge to an existing closed-form solution and some of which do not. Pivotal parameters, rules, or interaction criteria are identified, allowing the original theory to be extended or modified to encompass the new findings
Experimental	A simulation is shown to converge to a corresponding model with a known closed-form solution. An extended (generalized) model with no closed-form solution is defined and its corresponding simulation converges an outcome that is different from the initial model and simulation. The outcomes of the respective simulations are compared to provide a proxy comparison of the respective models

that higher consumer price sensitivity, which leads to more competitive pricing by firms, also leads to greater variance in the distribution of prices about their Nash equilibrium. One could explore whether and under what conditions the magnitude of price variations formally or systematically relate to other variables in the model. Third, one could relax the assumption that $p_t^{\mathrm{e}} = p_0$ in Eq. 10 so that households modify their expectations for inflation as well as income. Fourth, the current model follows the New Keynesian assumption of sticky prices (wages) in labor markets, in which case firms adjust their quantity of labor. Future work can analyze the importance of this assumption by relaxing it and seeing how it affects prices in both labor and goods markets. Fifth, one could relax the assumption of fixed interest rates to broaden the definition of Nash equilibrium and allow firms to pursue financial returns from a price-based, in addition to quantity-based, perspective. Sixth, one could explore the effects of cross-sectional variation in agent characteristics, such as households' discount rates or firms' production rates. Seventh, the simulation is ideal for introducing and observing the cross-market life-cycle effects of a population bulge or dip.

Additionally, agent rules could be designed in accordance with various explicit models of confidence (Batchelor & Dua 1998; Bram & Ludvigson 1998; Desroches & Gosselin 2002; Garner 2002) to explore potential responses to hypothetical economic shocks. One could explore the role of leisure in consumption profiles (Heckman 1974; Becker and Ghez 1975; Bullard and Feigenbaum 2004), precautionary savings (Hubbard, Skinner, & Zeldes 1994; Carroll 1997, Abel 1985; 2001; Poterba 2001; Wang 2004), and trade between multiple interactive economies (Sayan and Uyar 2002). These and other extensions of the mechanics of firm and household behavior in markets could give important new insight into Nash equilibria.

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