Solutions for Go Figure 2004

1. (a) 3. 20 minutes is $\frac{1}{3}$ hour, since there are 60 minutes in an hour and $\frac{20}{60} = \frac{1}{3}$. Alice paints 1 door per $\frac{1}{3}$ hour, so her speed (painting rate) is

$$\frac{1 \text{ door}}{\frac{1}{3} \text{ hour}} = 3 \text{ doors/hour.}$$

(b) $\frac{3}{4}$. 80 minutes is 4/3 hour, since there are 60 minutes in an hour and $\frac{80}{60} = \frac{4}{3}$. Barbara paints 1 door per 4/3 hour, so the number of doors she can paint per hour is

$$\frac{1 \text{ door}}{\frac{4}{3} \text{ hour}} = \frac{3}{4} \text{ doors/hour.}$$

(c) $\frac{15}{4}$ or $3\frac{3}{4}$. Alice can paint 3 doors per hour and Barbara can paint $\frac{3}{4}$, so working together they can paint $3 + \frac{3}{4} = \frac{15}{4}$.

(d) 8. They paint $\frac{15}{4}$ doors/hour and must paint 30 doors, so they need

$$\frac{30 \text{ doors}}{\frac{15}{4} \frac{\text{doors}}{\text{hour}}} = 30 \text{ doors} \times \frac{4}{15} \frac{\text{hours}}{\text{door}} = 2 \times 4 \text{ hours} = 8 \text{ hours.}$$

2. (a) $A = 5$, $B = 2$, $C = 1$. First we look at possible values for A. 9 is too large since the second number is at least 700 and the first number would be at least 900 if $A = 9$. Therefore the product would be at least $700 \times 900 = 630000$, while the product is less than 500000. A cannot be $6, 7$, or 8 because those digits are already explicitly used in the problem. If $A = 4$, then the first number is less than 500, the second is less than 800 and the product is less than $500 \times 800 = 400000$. Since the product must be at least 408000, this is too small. Therefore $A = 5$, the only remaining choice since values less than 4 are also too small. Now consider the units digit. We use the notation $u(n)$ to represent the units digit of number n. The units digit of the product is equal to the units digit of $6 \times B$. This must be equal to B. It's easiest to find the possible values for B by making a table:

B:	0	1	2	3	4	5	6	7	8	9
$6B$:	0	6	12	18	24	30	36	42	48	54

From this table, we see $u(6B) = B$ for half the possible values of B: $0, 2, 4, 6$ and 8. However $4, 6$ and 8 are already used as explicit digits. If $B = 0$, then the product would be $AA6 \times 7A \times 10$. So the tens digit of the product (C) must be equal to the units digit of $6A = 6 \times 5 = 30$. However, if $C = 0$, then B and C are the same digit, and the problem states that they are different. Therefore $B = 2$. Now we know the two multiplicands: 556 and 752. Their product is 418112, which has the correct form if $C = 1$. Note: another way to eliminate $B = 0$ as a possibility is to try the product $556 \times 750 = 417000$. This product does not have a consistent value for C (The first appearance of C must be a 1 and the others must be 0).

(b) $A = 0$, $B = 7$, and $C = 1$. Cross multiplying, we have $BAA \times 9C = 63BA \times CA$. The units digits of each of these products must be equal. Using the notation $u(n)$ for the units digit of number n, we have $u(AC) = u(A * A)$. Let's consider the possible values of A paired with C based only on the requirement that $u(AC) = u(A * A)$ and that A and C must differ from the digits 3, 6, and 9 used explicitly in the problem. We could have $A = 0$ combined with any value of C not in the set of used digits. Since $0 \times 0 = 0$ and $0 \times C = 0$ for any C. We cannot have $A = 1$ because $1 \times 1 = 1$ and $1 \times C = C \neq 1$ for any $C \neq 1$. If $A = 2$, then $A * A = 4$ and the only possible value of C is 7 (since $2 \times 7 = 14$, which has a units digit of 4, and no other value of $C \neq 2$ has $u(AC) = 4$). A cannot be 3, because 3 is used explicitly in the problem. A cannot be 4 because $u(4 \times 4) = 6$ and the only possible value of C is 9 (forbidden because 9 is used in the problem). $A = 5$ is possible with $C = 1$ or $C = 7$ (the other odd values are used explicitly in the problem). $A = 6$ is not allowed because 6 is a forbidden (used) digit. $A = 7$ is not allowed because there is no $C \neq 7$ such that $u(7C) = 9$. $A = 8$ is not allowed because the only possible value of $C \neq 8$ such that $u(8C) = 4$ is 3, which is a forbidden (used) digit. Finally $A = 9$ isn't allowed because 9 is used. In summary, our possible (A, C) pairings at this point are $(2, 7), (5, 1), (5, 7)$, or $A = 0$ and C any unused digit (anything but $0, 3, 6$ or 9). Now consider the first fraction. The largest value of B is 8 so the numerator as at most 900. The denominator is at least 6300. Therefore the first fraction is smaller than $\frac{900}{6300} = \frac{1}{7}$. Thus we must have $\frac{CA}{9C} < \frac{1}{7}$. Using the (A, C) pair $(2, 7)$, we have $\frac{CA}{9C} = \frac{72}{97} > \frac{1}{7}$, so this pair is not possible. Similarly for the next two pairs, $\frac{15}{91} > \frac{1}{7}$ and $\frac{75}{97} > \frac{1}{7}$. Therefore $A = 0$, and we require a value of C such that $\frac{C0}{9C} < \frac{1}{7}$. This is true for $C = 1$, but is not true for $C = 2$ or any larger values of C. Therefore $C = 1$. Now we must find a B such that $\frac{B00}{63B0} = \frac{10}{91}$. Since both the numerator and denominator of the lefthand fraction are divisible by 10, we can simplify the problem by dividing both numerator and denominator by 10. Thus we must find a B such that $\frac{B0}{63B} = \frac{10}{91}$. Therefore, by cross multiplication, we have $91 \times B0 = 63B0$.

$$\begin{aligned} 91 \times B0 &= (90 \times B0) + (1 \times B0) \\ &= (9 \times B00) + B0 \\ &= (9B \times 100) + B0. \end{aligned}$$

So we require that $9B \times 100 + B0 = 63B0$. Thus $9B = 63$ so $B = 7$.

3. (a) 347. In this progression, the terms are 8×1, 8×2, 8×3 and so on. The ith term is $8 \times i$. Since $2776 \div 8 = 347$, then $2776 = 8 \times 347$ and there are 347 terms.

   (b) 1392. Since 347 is odd, there is a single middle element. Since $347/2 = 173$ with a remainder of 1, the 174th term is the middle one (there are 173 that preceed it and 173 that follow). The 174th term is $8 \times 174 = 1392$.

   (c) 347. This progression has the same difference between terms (8) as the progression in part a. Each term is the same as the corresponding term in part a reduced by 28. That is, $-20 = 8 - 28$, $2748 = 2776 - 28$, etc. Since every term in the first progression has a corresponding term in this one and they pair exactly, this progression has the same number of terms as that in part a.

   (d) 1364. Because the terms of this progression are in one-to-one correspondance with the terms in the progression in part a, the middle term is equal to the middle term of the first progression minus the difference between the progressions, which is 28. The middle term of the progression of part a is 1392 (from part b). Therefore the middle term of this progression is $1392 - 28 = 1364$.

   (e) 1364. The first term is -20, the last is 2748. The arithmetic average is $\frac{2748-20}{2} = \frac{2728}{2} = 1364$.

   (f) 1364. The second term (-12) is 8 larger than the first term, but the next to last (2740) is 8 smaller than the last. Therefore the sum of the terms is the same (2740), as for the first and last pair. Therefore, they have the same average value.

   (g) 473308. The pattern shown in the last two parts continues: each pair created by moving one higher from the bottom of the progession and one down from the top (for example, 2nd term and second from last) has the same average. There are 347 terms, each with an average value of 1364 when paired with another appropriate term as necessary (the middle term is already the average value). Therefore the sum of all the numbers is the number of terms times the average value: $347 \times 1364 = 473308$.

(h) 691. Since there are 347 unique numbers in the progression, there are $347 - 1 = 346$ ways to pair the first term (-20) with another term from the progression. The resulting sums (pairing in order) are $-32, -24, -16, -8, 0, 8 \ldots, 2728$. Pairing the second term (-12) with higher terms only, the sums are $-16, -8, 0, \ldots, 2736$. Every term is represented in the sequence before except for the last. Therefore, we generate one additional number. This pattern repeats: to obtain the set of sums from the previous set, drop the first two terms, and add a new term at the end 8 larger than the previous last term. Starting with the second term and ending with the next-to-last term (there is nothing to pair the last term with at the end), there are 345 such sets of sums, each adding one more sum. Therefore, overall, there are $346 + 345 = 691$ unique sums of pairs of terms from the progression of part c.

4. (a) $121, 1331$, and 14641. $11 \times 11 = 121$, $11 \times 11 \times 11 = 1331$, and $11 \times 11 \times 11 \times 11 = 14641$.

   (b) 1. 11^x has a units digit of 1 for any x because all the multiplicands in the product have a units digit of 1.

   (c) 6. This question is asking for the tens digit of 11^{1386}. Because the units digit is 1, $11^{1386} - 1$ must have a units digit of 0. Thus is it divisible by ten. The new units digit after this division is the tens digit of 11^{1386}. The tens digit of 11^x for any number x is equal to the units digit of x. To see this, consider any number with a units digit of 1. Let y represent the tens digit of this number. We can ignore any more significant digits because they will not effect the tens digit of the product. Now consider the product $y1 \times 11$.

$$\begin{array}{ccc} & y & 1 \\ \times & 1 & 1 \\ \hline & y & 1 \\ y & 1 & 0 \\ \hline \end{array}$$

   The tens digit of the product is always the units digit of $(y + 1)$. That is, the tens digit is always $y + 1$, except that if $y = 9$, then the tens digit is zero. For those familiar with modular arithmetic, the tens digit is $(y + 1)\text{mod}10$. To compute 11^2, we have $y = 1$ and the tens digit of the product is $1 + 1 = 2$. Multiplying by 11 again (to compute 11^3) increases the tens digit of the product from 2 to 3 . This pattern continues with the tens digit cycling back to 0 after 9. Thus the tens digit of 11^x always matches the units digit of x. For this problem, $x = 1386$, so the tens digit of $11^{1386} = 6$.

5. (a) 4. $2 \times 2^3 = 2 \times 2 \times 2 \times 2 = 2^4$.

   (b) 5. In a number of the form y^x, y is called the base and x is called the exponent. As part a demonstrates, the product of y^a and y^b is y^{a+b}. The number of times we multiply the base by itself is the sum of the number of times for each multiplicand.

   (c) 10. $2 = 2^1$. So the sum of the exponents is $1 + 4 + 5 = 10$.

   (d) 480248. The exponent x is the sum of the exponents of the multiplicands. This is $0+8+16+\ldots+2768$. The terms of the progression are the same as those in question 3c except that each term is 20 larger. Since there are 347 terms (by question 3c), the sum of all the terms is $347 \times 20 = 6940$ larger than the sum we calculated for question 3g. Therefore $x = 6940 + 473308 = 480248$.

6. (a) 20. There are 4 colorings that use only one color (all balls with the same color), one for each of the 4 colors. If the coloring uses 2 colors, then two balls have one color (color a) and one ball has a different color (color b). There are 4 ways to choose color a. Once we have chosen color a, there are 3 remaining choices for color b. Therefore, there are $4 \times 3 = 12$ colorings that use 2 colors. If the coloring uses 3 colors, then each of the colors that's used colors one ball. There

are 4 ways to choose the color that's *not* used. Therefore there are 4 colorings that use 3 colors. Since we can only use 1, 2, or 3 different colors, the number of colorings is $4 + 12 + 4 = 20$.

(b) 20958500. There are 500 colorings that use only one color (all balls with the same color), one for each of the 500 colors. If the coloring uses 2 colors, then two balls have one color (color a) and one ball has a different color (color b). There are 500 ways to choose color a. Once we have chosen color a, there are 499 remaining choices for color b. Therefore, there are $500 \times 499 = 249500$ colorings that use 2 colors. If the coloring uses 3 colors, then each of the colors that's used colors one ball. There are $\binom{500}{3}$ ways to choose the three colors we use. Therefore there are $\frac{500\times499\times498}{3\times2} = 500 \times 499 \times 83 = 20708500$ colorings that use 3 colors. Since we can only use 1, 2, or 3 different colors, the number of colorings is $500 + 249500 + 20708500 = 20958500$.

7. (a) Any three-digit number that has a units digit of 4 or 9 (such as 664 or 129). A number is divisible by 5 if and only if the units digit is 0 or 5. Therefore $K^2 - 1$ is divisible by 5 if and only if K^2 has a units digit of 1 or 6. For any number K (three-digit or otherwise), the units digit of K^2 is determined by the units digit of K. More precisely, if we let $u(n)$ be the units digit of number n, and we let K_0 represent the units digit of K, then we have $u(K^2) = u(K_0^2)$. We look at the possible values for $u(K_0)$ in a table:

$u(K_0)$:	0	2	3	4	5	7	8	9
$u(K_0^2)$:	0	4	9	6	5	9	4	1

Therefore as long as the units digit of K is either 4 or 9, then $K^2 - 1$ is a multiple of 5.

(b) 2571. Because $K + 1$ is not a multiple of 7, we can express K as $7n + r$ for an integer n and a remainder $r \in \{0, 1, 2, 3, 4, 5\}$. That is, we know $r \neq 6$ because adding 1 to such a number will create a number that is divisible by 7. Now consider K^3. When using the distributive property to compute K^3, only computations involving the remainder are not divisible by 7. For example, if $K = 72 = 10 \times 7 + 2$, then $K^2 = (10 \times 7 + 2)(10 \times 7 + 2) = 100 \times 7^2 + 40 \times 7 + 2 \times 2$. Every term in this sum is multiplied by 7 except the last. Therefore, $K^3 + 1$ is divisible by 7 if and only if $r^3 + 1$ is divisible by 7. The possible choices for r are those listed above. Computing these in a table:

r:	0	1	2	3	4	5
r^2:	0	1	4	9	16	25
r^3:	0	1	8	27	64	125
$r^3 + 1$:	1	2	9	28	65	126

We see that of these options, only $r = 3$ and $r = 5$ have $r^3 + 1$ divisible by 7. This is all that's required. So a four-digit number has the properties we require as long as it has a remainder of 3 or 5 when divided by 7. Therefore 2 out of every 7 consecutive numbers has the property. Looking at four-digit numbers, the first four-digit number 1000 has a remainder of 6 when divided by 7. Therefore, the first four-digit number divisible by 7 is 1001. The last four-digit number divisible by 7 is 9996. There are $\frac{9996-1001}{7} = 1285$ groups of 7. Within these groups, there are $1285 \times 2 = 2570$ with a remainder of 3 or 5 when divided by 7. Finally, the number 9999 has a remainder of 3 when divided by 7. So the total number is $2570 + 1 = 2571$.

8. $1, 1, 1, 1, 1, 1, 1, 1, 1, 1$. For this strategy the amount of pie you acquire by the end of the game is $\frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1$. Suppose we decided to instead send away two people in the first round (then finish with one person per round). The effect is to replace the $\frac{1}{9}$ pie acquired in the second round by an extra $\frac{1}{10}$ pie in the first round (and then finish with 8 more rounds as before). However, $\frac{1}{9} > \frac{1}{10}$, so it would be better to delay the departure of the second person. In general, any

round where you start with p people and send away $x > 1$ people results in your accepting $x-1$ pieces of size $\frac{1}{p}$ instead $x-1$ of pieces that will be at least as big as $\frac{1}{p-1}$.

9. 7. The following simple placement has a gap of 7:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

The difference between any pair of horizontal neighbors is 1. Vertical neighbors have a difference of 6. The neighbors on the diagonals going bottom left to top right have a difference of 4. And neighbors on the other diagonals have a difference of 7.

To see this is the best possible gap value, think of the boxes as a drawing on the floor large enough so you can stand in any box. Suppose you are allowed to move from one box to another provided the two boxes are neighbors. Notice that for any two boxes out of the set of 36, you can get from one to the other in at most 5 moves (that's enough to get from the first to the last row, to get from the top left corner to the bottom right along the diagonal, etc). The numbers 1 and 36 must appear in some pair of boxes. There is a path with at most 5 moves between this pair. Therefore, the value must climb from 1 to 36 in at most 5 steps. Therefore the average difference in value between the neighbors we use for these moves is at least $(36-1)/5 = 7$. At least one of these neighbor pairs, therefore, must have a value difference of at least 7.

10. and 11.

(a) 2. The positive multiples of 2 produce all even numbers starting with 2: $2, 4, 6, 8, \ldots$. We can produce the number 3 as 1×3. Then we can produce all other odd numbers starting with 5 by adding a positive multiple of 2. So we can produce all even and all odd numbers starting with 2. But we cannot produce the number 1.

(b) 8. We will put all numbers into three groups based upon their remainder when divided by 3. There are only three possible values for the remainder when dividing by three: $0, 1$, or 2. Numbers that are multiples of 3 have a remainder of 0 when divided by three. The numbers $1, 4, 7, 10, \ldots$ have a remainder of 1 when divided by 3 and the numbers $2, 5, 8, 11, 14, \ldots$ have a remainder of 2 when divided by 3. The set $\{3, 5\}$ can produce all multiples of 3 (which have a remainder of 0 when divided by 3) starting with 3. The first number with a remainder of 2 that the set can produce is 5. But by adding any multiple of 3 to 5, it can produce any number with a remainder of 2 when divided by 3 starting with the number 5. Similarly, the first number the set can produce that has a remainder of 1 when divided by 3 is 10, and it can then produce all such numbers after that. So we can produce all the numbers with remainder $0, 1$, and 2 starting with $3, 10$, and 5 respectively. So the horizon is no larger than 10. The largest number with remainder 1 that isn't produced is 7, so the horizon is at least 8. The set can produce 8 since it has a remainder of 2 and is larger than 5 and the set can produce 9 since it is a multiple of 3 larger than 3. Therefore, the set can produce all integers greater than or equal to 8.

(c) None. The set $\{21, 35\} = \{3 \times 7, 5 \times 7\}$. So all numbers it produces are multiples of 7. So it will never produce a number that has a remainder when divided by 7. Since there are an infinite number of these, the set has no horizon.

(d) 56. The set $\{21, 35\} = \{3 \times 7, 5 \times 7\}$. So the set $\{21, 35\}$ produces a number $n = 7 \times k$ if and only if set $\{3, 5\}$ produces the number k. From part (b) of this problem we know the set

$\{3, 5\}$ produces all integers greater than or equal to 8. So Set $\{21, 35\}$ produces all multiples of 7 greater than or equal to $8 \times 7 = 56$.

(e) 182. From part (d), we know that the set $\{21, 35\}$ produces all multiples of 7 starting with 56. It doesn't produce any number that is not a multiple of 7 and it doesn't produce some smaller multiples like 49. If set $\{21, 22, 35\}$ has a horizon, then from that horizon onward, it must be able to produce numbers with every possible remainder when divided by 7 (there are 7 possibilities: $0, 1, 2, 3, 4, 5,$ and 6.) To make the rest of this solution easier to read, we will introduce some terminology. If a number n has a remainder of r when divided by seven we say "n is (equal to) r mod 7." 22 is 1 mod 7. So the smallest number that is equal to 1 mod 7 that this set can produce is 22. The smallest number equal to 2 mod 7 is $22 \times 2 = 44$. In general, the remainder of $22 \times k$ when divided by 7 for any integer $k \geq 1$ is the same as the remainder when k is divided by 7. So the first number the set can produce that is equal to 3 mod 7 is $3 \times 22 = 66$ and so on. Finally, the smallest number equal to 6 mod 7 that this set can produce is $6 \times 22 = 132$. By adding multiples of 21 and 35 the set can then produce *all* numbers equal to 1 mod 7 starting at $22 + 56 = 78$. It can produce all numbers equal to 2 mod 7 starting at $44 + 56 = 100$ and so on. Finally, it can produce all numbers equal to 6 mod 7 starting with $132 + 56 = 188$. So the horizon is at most 188, but the set can produce all the numbers moving backward from 188 until the first number equal to 6 mod 7. This is $188 - 7 = 181$. Therefore the set can produce all numbers greater than or equal to 182. But it cannot produce 181. To see this, recall that the smallest number equal to 6 mod 7 that the set can produce is 132. The next smallest number equal to 6 mod 7 that is produced by 22 alone is 264. Therefore, the set $\{21, 22, 35\}$ produces 181 if and only if the set $\{21, 35\}$ produces $181 - 132 = 49$. The set $\{21, 35\}$ does not produce 49 because the set $\{3, 5\}$ does not produce 7.

(f) 152462. The set $\{359, 5943, 6226, 9905\} = \{359, 21 \times 283, 22 \times 283, 35 \times 283\}$. 283 is prime. We can check this by noticing that $17^2 = 289 > 283$, so we only have to verify that 283 is not divisible by any prime number up to 13. We can see that by inspection for $2, 3, 5,$ and 11 (using the fact that a number is divisible by 3 if and only if the sum of its digits is divisible by 3 and noting that any multiple of 11 has a units digit of 1). We can then verify neither 7 nor 13 divides 283. Any number produced by the subset $\{5943, 6226, 9905\}$ is a multiple of 283. In particular, this set produces all numbers produced by $\{21, 22, 35\}$ multiplied by 283. From part (e), that means the subset $\{5943, 6226, 9905\}$ produces all multiples of 283 starting with $182 \times 283 = 51506$, but it cannot produce $181 \times 283 = 51223$. If the full set has a horizon, it must produce numbers with all possible remainders when divided by 283 (not just those with remainder 0), since the first 283 numbers after the horizon will together have all possible remainders. Any number this set produces that isn't 0 mod 283 must be a multiple of 359, call it x, with added multiples of 283 and it's remainder mod 283 will be the same as x's remainder mod 283. The first number the set can produce that is not a multiple of 283 (is not 0 mod 283) is 359. Consider the set of numbers of the form $359n$ for $n = 0, 1, 2, \ldots, 282$. No pair of numbers in this set has the same remainder when divided by 283. To see this, suppose two numbers $359n_1$ and $359n_2$ are equal mod 283 with $282 \geq n_2 > n_1$. Then $359(n_2 - n_1)$ must be a multiple of 283, say $283n_3$. So $359 = \frac{283n_3}{n_2 - n_1}$. But $(n_2 - n_1)$ doesn't divide 283, since it's greater than 1 and less than 283 and 283 is prime. If we think of the times table for 359 as repeatedly adding 359 to the previous entry, then each time we move up in the table, we pass at least one multiple of 283, but no more than 2 multiples of 283 (because $283 < 359 < 2 \times 283$). Thus we know that $(n_2 - n_1) < n_3 < 2(n_2 - n_1)$. Therefore $(n_2 - n_1)$ doesn't divide n_3 either. Thus we know that the each of the first 283 entries in the multiplication table for 359 (starting at zero) will have a different remainder mod 283. Therefore, $282 \times 359 = 101238$ is the first number produced by this set that is equal to 207 mod 283 (the actual value of the remainder doesn't matter). Therefore, using an argument similar to the one above, the set can generate all numbers equal to 207 mod 283 starting with $101238 + 51506 = 152744$. This is the first number with the appropriate remainder plus the smallest multiple of 283 after which we can generate all such multiples. It can't generate the immediately preceeding number with a remainder of 207, which is $152744 - 283 = 152461$. So the set can produce every number starting with 152462.

12. (a) 12. Thoughout this solution, we use the name of a segment (like AB) to refer to both the segment itself (in discussions) and the length of the segment (in computations). To compute the length of BF, we construct segment BI perpendicular to GD as shown in the figure. First we'll show that triangles labeled 1 and 2 are similar. Angles GBE and ABC are equal because they are opposite angles made by a pair of intersecting lines. However, because $AB = AC$, triangle ABC is isosceles, so angles ABC and ACB are also equal. Therefore $GBE = ACB$. The triangles also both have right angles. Since they have two pairs of equal angles, they are similar. Triangle 2 is also similar to triangle 3. Angles IGB and FBC are equal because they are corresponding angles made by a line crossing a pair of parallel lines. (Note: one could also show these triangles are similar by first showing triangle 2 is similar to triangle DGC. This will not require the parallel-line argument, other than realizing that angle GDC must be a right angle). Triangles 2 and 3 also both have right angles, so they have two pairs of equal angles. Therefore triangles 1 and 3 are similar. In fact, since they share a corresponding side GB, as the hypotenuse in both cases, triangles 1 and 3 are congruent (equal). Since we were given that $GE = 4$, we now have that $GI = 4$. Therefore $DI = GD - GI = 16 - 4 = 12$. Quadrilateral $IDFB$ is a rectangle since a pair of opposite angles are both right angles. Therefore $DI = BF = 12$.

(b) $\frac{7}{3}$. We are given that $AB = 13$ and we computed $BF = 12$ in part a. Applying the Pythagorean theorem to right triangle ABF, we have $(AF)^2 + (12)^2 = (13)^2$. So $AF^2 = 169 - 144 = 25$. Therefore $AF = 5$. We were given that $AC = AB = 13$, so $FC = AC - AF = 13 - 5 = 8$. Now we use the similarity of triangles 2 and 3. This gives the ratio $\frac{BF}{FC} = \frac{GI}{BI}$. Filling in values: $\frac{12}{8} = \frac{4}{BI}$, so $12 \times BI = 32$ and we have $BI = \frac{8}{3}$. Again using rectangle $IDFB$, we have $BI = DF = \frac{8}{3}$. Finally $AD = AF - DF = 5 - \frac{8}{3}$, so $AD = \frac{7}{3}$.