Solutions for Go Figure 2003

1. (a) \$900. The investment decreased by 10% in two years, so it then has only 90% of its original value. $1000 \times .9 = 900$.

   (b) \$902.50. The investment dropped 5% in its first year to a value of $1000 \times .95 = 950$. This now decreases by 5% of the smaller amount: $950 \times .95 = 902.50$.

2. (a) $A = 2$, $B = 8$, and $C = 5$. The units digit of the product is equal to the units digit of $3 \times A$. There is only one single-digit number that has a units digit of 6 when multiplied by 3. That is 2 ($3 \times 2 = 6$). Therefore $A = 2$. Therefore the first number is 723. Now consider the tens digit of the product. This is equal to the sum of the tens digit of 723×2 (which is 4) plus the tens digit of $723 \times B \times 10$ (which is the units digit of $3 \times B$.). So we must find a B such that the units digit of $3 \times B + 4$ is B. The easiest way to find such a B is to make a table. We use the notation $u(n)$ to represent the units digit of number n.

B:	0	1	2	3	4	5	6	7	8	9
$3B$:	0	3	6	9	12	15	18	21	24	27
$u(3B+4)$:	4	7	0	3	6	9	2	5	8	1

   There are two columns in the table where the last entry is equal to the first: 3 and 8. However, 3 is already used explicitly in the problem, so the only possible value for B is 8. We determine C by multiplication: $723 \times 82 = 59286$, which has the correct form with $C = 5$.

   (b) $A = 1$, $B = 5$, $C = 6$. Looking at the first equality, by cross multiplication we have $A39 \times CB = C9B \times A3$. The units digit of each of these products must be equal, so again denoting the units digit of a number n by $u(n)$, we have $u(9 \times B) = u(3 \times B)$. Again, it is easiest to make a table (or add another row to the table you made for the first part of this problem):

B:	0	1	2	3	4	5	6	7	8	9
$3B$:	0	3	6	9	12	15	18	21	24	27
$9B$:	0	9	18	27	36	45	54	63	72	81

   There are two possible values for B where $u(3 \times B) = u(9 \times B)$, namely 0 and 5. However, $B \neq 0$ because division by 0 is illegal. (For more advanced students, even if you wish to consider division of a finite number by zero to be equal to infinity in the limit, certainly the other two fractions are finite so $B = 0$ is not correct). Therefore, we have $B = 5$. Now the second equality becomes $\frac{A3}{C5} = \frac{A}{5}$. Again by cross-multiplication we have $A3 \times 5 = C5 \times A$. The units digits of these two products much match so we have $u(5 \times A) = 5$. Therefore A is odd. Futhermore, we know A is not equal to $3, 5$, or 9 because these digits are already used. Therefore A is either 1 or 7. Consider the second equality and observe that both $A3$ and A are integers. Therefore, A must be a factor of $A3$. We then know that $A = 1$ because 1 is a factor of 13 but 7 is not a factor of 73. Now the final equality becames $\frac{13}{C5} = \frac{1}{5}$, so $C5 = 65$ giving us $C = 6$. As a final check, we verify that $\frac{139}{695} = \frac{1}{5}$ because $139 \times 5 = 695$.

3. (a) 5. Since 510, 50 and 20 are all divisible by 10, this is the same problem as how many ways can you create 51 from only sums of 5's and 2's. Since 51 is odd, there must always

be an odd number of 5's, with a maximum of 9: $9 \times 5 + 2 \times 3 = 51$. One can always replace two 5's with five 2's since $2 \times 5 = 5 \times 2$. Therefore, we can create 51 with $1, 3, 5, 7$, or 9 fives (and an appropriate number of twos).

(b) \$1.19. There can be no dollar coins, since these have value exactly \$1. There can be at most one half dollar, since two would again sum to a \$1. Since two quarters are equal in value to a half dollar, without loss of generality (with respect to maximum value), we can assume the box contains at most one quarter. Similarly, since 5 dimes are equivalent to a half dollar, we can assume the box has at most 4 dimes. Since 2 nickels are equivalent to a dime, we can assume the box has at most 1 nickel, and finally, since 5 pennies are equivalent to a nickel, we can assume the box has at most 4 pennies. Since two dimes and a nickel is equivalent to a quarter, we cannot have all three coins in the box (we'd just use a quarter instead). Therefore, we either must restrict the box to hold only one dime, or restrict it to hold no nickels. The second choice (drop 1 nickel) removes less value than removing 3 dimes. At this point, our box contains at most 1 half dollar, 1 quarter, 4 dimes, and 4 pennies for a total of \$1.19. We confirm there is no way to create exactly \$1 from this set. Any set of coins of value \$1 would have to include the quarter and half dollar (otherwise, there wouldn't be enough value). However, there is no way to create 25 cents from four dimes and four pennies (3 dimes is too much; 2 dimes plus all the pennies isn't enough).

4. 87 and 88. Two consecutive pages in a book will have consecutive numbers, say n and $n + 1$. Their product will be close to a perfect square. Since $100 \times 100 = 10000$, we know the page numbers are less than 100. Squares of numbers like 50 and 60 are easy to compute (the square of the tens digit times 100), so one can see that the page numbers we seek are between 80 and 90 since $80^2 = 6400$ and $90^2 = 8100$. The units digit is determined by the product of two consecutive integers. There are only two such products that give a units digit of 6, namely $2 \times 3 = 6$ and $7 \times 8 = 56$. Thus the pair of pages is either 82 and 83 or 87 and 88. Checking both possibilities, the second pair $(87, 88)$ gives the right product 7656.

5. (a) 295. In this progression, the terms are 5×1, 5×2, 5×3, and so on. The 59th term is $5 \times 59 = 295$.

(b) 63. The sequence is an arithmetic progression with a fixed difference of 5. If this sequence had begun with 5, we would only have to divide the final term by 5 to find the number of terms. But we can compare sequences with the same fixed difference and length. Therefore, the sequence $5, 10, 15, 20 \ldots, 315$ (where each term is 7 less that the corresponding term in our sequence) has the the same number of terms as our sequence. So the number of terms is $315/5 = 63$.

A "classic" method to determine the number of sequence elements between a start value and an end value is to compute the span of the sequence (difference between the first and last term): $322 - 12 = 310$. Dividing by the difference between successive terms (in this case 5) gives us the number of terms *after the first one*: $310/5 = 62$ terms after the first one. We must then add one for the first term, so there are 63 in total. You should use this method with caution since it is easy to forget to add one for the first term.

(c) $431 = 8 + 423$.

(d) 42. Using the method described for part (b), we compute the number of terms in the arithmetic progression $8, 13, 18, \ldots, 423$. It is the same as the number of terms in $5, 10, 15, \ldots, 420$, which is $420/5 = 84$. The first and last pair 8 and 423 sum to 431. The second term 13 is 5 more than the first term. If it is paired with the next-to-last (83rd) term, which is 5 fewer than 423, we have both added and subtracted 5 compared to the first pair, and therefore has the same sum of 431. Is we continue to pair the ith smallest with the ith largest, we always have the same sum. Since we have an even number of terms, there are $84/2 = 42$ pairs.

(e) 18102. There are 42 pairs and each pair sums to 431. Therefore the sum of the terms in the arithmetic progression equals the sum of all the pairs equals $42 \times 431 = 18102$.

(f) 44. $446 = 423 + 23$. This is the sum of the last term and the fourth term. We can also create a sum of 446 by pairing the 5th term 28 with the next-to-last term 418 and so on. This is the same sort of pairing we did in part (d) except that, once we remove the initial 3 terms from the pairing process, we have an odd number of terms. The middle term in this process 223 is exactly half of 446 and it has no mate (it would have to be paired with itself). Therefore, there are $80/2 = 40$ pairs that sum to the forbidden value of 446. We can select only one number from each such pair. We can also select 223 and the first three terms for a total of $40 + 4 = 44$.

6. (a) $a = 3$, $b = 81$. The ratio is $\frac{27}{9} = 3$, so $a = 1 \times 3 = 3$ and $b = 27 \times 3 = 81$.
   (b) $c = \frac{27}{2}$, $d = \frac{81}{4}$. The ratio is $\frac{6}{4} = \frac{3}{2}$. Therefore $c = 9 \times \frac{3}{2} = \frac{27}{2}$ and $d = \frac{27}{2} \times \frac{3}{2} = \frac{81}{4}$.
   (c) $e = 1$, $f = \frac{1}{2}$. The ratio is $\frac{4}{8} = \frac{1}{2}$. Therefore $e = 2 \times \frac{1}{2} = 2/2 = 1$ and $f = 1 \times \frac{1}{2} = \frac{1}{2}$.
   (d) $g = 27$, $h = \frac{81}{4}$. The ratio is $\frac{36}{48} = \frac{3}{4}$. Therefore, $g = 36 \times \frac{3}{4} = 27$ and $h = 27 \times \frac{3}{4} = \frac{81}{4}$.

7. (a) 4, 8, 16, 32. $S(1) = 1 + 2 = 3$ so $S(1) + 1 = 4$. $S(2) = S(1) + 2^2 = 3 + 4 = 7$. $S(3) = S(2) + 2^3 = 7 + 8 = 15$. $S(4) = S(3) + 2^4 = 15 + 16 = 31$.
   (b) 992.

$$
\begin{aligned}
2^5 + 2^6 + 2^7 + 2^8 + 2^9 &= 2^5(1 + 2 + 2^2 + 2^3 + 2^4) \\
&= 32 \times S(4) \\
&= 32 \times 31 \\
&= 992.
\end{aligned}
$$

   (c) 2048. $S(10) + 1 = S(4) + 1 + (2^5 + 2^6 + 2^7 + 2^8 + 2^9) + 2^{10}$. From part (a) we know that $S(4) + 1 = 32$. From part (b) we know that $2^5 + 2^6 + \ldots + 2^9 = 992$. $2^{10} = 1024$ so $S(10) + 1 = 32 + 992 + 1024 = 2048 = 2^{11}$. You may have noticed $S(n) + 1 = 2^{n+1}$ as a pattern from the first four parts of this question. This is true in general from the following argument. $S(n) = 1 + 2 + \ldots + 2^n$. Multiplying both sides by 2, we have $2 * S(n) = 2 + 4 + \ldots + 2^{n+1}$. Subtracting the first equality from the second we get:

$$
\begin{array}{rrclll}
 & 2 * S(n) & = & & 2 + 4 + \ldots + 2^n & +2^{n+1} \\
- & S(n) & = & 1+ & 2 + 4 + \ldots + 2^n & \\
\hline
 & S(n) & = & -1 & & +2^{n+1}
\end{array}
$$

8. A number is prime if it is only divisible by itself and 1. The prime factorization of a number n represents n as the product of prime numbers. That is, it represents n as the product of some number of 2's, some number of 3's, and so on through all the prime factors of n. For example

$36 = 2^2 3^2$. Suppose n (such as 36) is a multiple of 9 but not a multiple of 27. Then the exponent of 3 in its prime factorization is 2. A factor f of n can be divisible by 3 or 9 (or not even by 3), but it cannot be divisible by any power of 3 higher than 3^2. The greatest common divisor (GCD) of two numbers n and m must be a factor of each of them. Thus the exponent of 2 in the prime factorization of $\mathrm{GCD}(n, m)$ can be at most the minimum exponent of 2 in the prime factorizations of n and m. The largest common divisor will have the exponent equal to this minimum for all its prime factors. For example

$$\mathrm{GCD}(2^x 3^y, 2^w 3^z) = 2^{\min(x,w)} 3^{\min(y,z)}.$$

The least common multiple of n and m must have the exponent of 2 *at least* as large as the exponent of 2 in either n or m. The smallest such multiple will have the exponent of 2 exactly equal to the maximum exponent of 2 in either n or m, and similarly, we can determine the exponent for all other prime factors by choosing the maximum corresponding exponent from n or m. For example

$$\mathrm{LCM}(2^x 3^y, 2^w 3^z) = 2^{\max(x,w)} 3^{\max(y,z)}.$$

We apply this to the first three subproblems.

(a) $\mathrm{GCD}(10, 15) = 5$, $\mathrm{LCM}(10, 15) = 30$. $10 = 2 \times 5$ and $15 = 3 \times 5$. The only common prime factor is 5, which appears once in each factorization. The LCM must have the maximum exponent for all factors appearing in the prime factorization, in this case an exponent of 1 for each of factor 2, 3, and 5, so $\mathrm{LCM}(10, 15) = 2 \times 3 \times 5 = 30$.

(b) $\mathrm{GCD}(12, 18) = 6$ and $\mathrm{LCM}(12, 18) = 36$. Since $12 = 2^2 \times 3$ and $18 = 2 \times 3^2$, $\mathrm{GCD}(12, 18) = 2 \times 3$ and $\mathrm{LCM}(12, 18) = 2^2 \times 3^2 = 36$.

(c) $a = 81$, $b = 10$, $c = 102$, $d = 33$, $e = 0$, $f = 0$, $g = 400$, $h = 56$, $i = 270$, $j = 86$, $k = 25$, $l = 2$. The exponents for GCD are the minimum exponent from either n or m. The exponents for LCM are the maximum from either n or m. A missing factor has an exponent of 0.

(d) 4. The factorization of 2100 is $2^2 \times 3 \times 5^2 \times 7$. Both numbers, say n and m must be divisible by 6, so each starts with 2×3 in their prime factorization. Between the two of them, they must also have exactly $2100/6 = 2 \times 5^2 \times 7$ as additional factors. If there were more factors, the LCM would be larger. We must assign each of these remaining factors to either n or m to find a pair satisfying the requirements of the problem statement (with the appropriate GCD and LCM). How many ways can we do this? The two fives must both be in one number or the other, otherwise the GCD would not be six. So any combination of $2, 5^2, 7$ can be multiplied by six to get one of the numbers, the other number is six multiplied by the remaining numbers. To determine the first number, we must make three choices, each with two options: include 2 as a factor or not, include 25 as a factor or not, and include 7 as a factor or not. Thus there are $2 \times 2 \times 2 = 2^3 = 8$ ways to pick a subset of $\{2, 25, 7\}$ to form the number n. The 8 choices are:

$$\{1, 2, 5^2, 2 \times 5^2, 7, 2 \times 7, 5^2 \times 7, 2 \times 5^2 \times 7\}$$

Each pair has been counted twice, for example $(6, 2001), (2001, 6)$. So there are actually $2^3/2 = 4$ pairs.

$$\{(6, 2100), (12, 1050), (150, 84), (42, 300)\}.$$

9. "Marvelous you solved the puzzle." In the following discussion, we will always enclose unresolved regions in a box and newly-resolved numbers will be in bold type. There are many ways to solve this problem. For example, the steps described here could be applied in different orders. There are probably other rules one could apply as well, all leading to the same result.

   Only numbers in the 10's and 20's are two-digit numbers so any digit not a 1 or 2 preceeded by a digit that is not a 1 or a 2 must be a single digit:

   $\boxed{2322417}$, **7**, $\boxed{2615}$, **4**, $\boxed{1127251018121313}$, **5**, $\boxed{2216}$, **6**, $\boxed{141921820}$, **9**

   Since $7, 4, 5, 6, 9$ are single digits, any other occurrence of these digits must be in the 10's or 20's:

   $\boxed{232}$, **24**, **17**, 7, **26**, **15**, 4, $\boxed{11}$, **27**, **25**, $\boxed{1018121313}$, 5, $\boxed{22}$, **16**, 6, **14**, **19**, $\boxed{21820}$, 9

   Zero is not a single digit, so if a zero occurs it must be with 10 or 20; in the first set of three digits, 232, 32 is not a feasible number, so the second 2 is the isolated digit 2. Since the single digit 2 is found, the remaining two's must be part of two digit numbers 23,21,22,12:

   23, **2**, 24, 17, 7, 26, 15, 4, $\boxed{11}$, 27, 25, **10**, $\boxed{18}$ **12**, $\boxed{1313}$, 5, **22**, 16, 6, 14, 19, **21**, $\boxed{8}$, **20**,9

   There is only one 13, so the block of 1313 must contain the single digit 1 (also the number 13 and the single digit 3), so any other 1's must be part of two digit numbers – 11,18; also the digit 8 is isolated.

   23, 2, 24, 17, 7, 26, 15, 4, **11**, 27, 25, 10, **18**, 12, $\boxed{1313}$, 5, 22, 16, 6, 14, 19, 21, **8**, 20,9

   Now the only ordering not known is the division of 1313. This could be either $13, 1, 3$ or $1, 3, 13$. These characters are VEL. Check the orderings of the message to see which fits. The only ordering that fits is $13, 1, 3$, so the permutation is:

   $$23, 2, 24, 17, 7, 26, 15, 4, 11, 27, 25, 10, 18, 12, 13, 1, 3, 5, 22, 16, 6, 14, 19, 21, 8, 20, 9$$

   and the message is:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
V	*A*	*E*	*U*	*D*	*E*	*E*	*Z*	*E*	*U*	*S*	*O*	*L*	*P*	*O*	*H*	*V*	*S*	*U*	*L*	*Z*	*T*	*M*	*R*	*O*	*L*	*Y*
M	*A*	*R*	*V*	*E*	*L*	*O*	*U*	*S*	*Y*	*O*	*U*	*S*	*O*	*L*	*V*	*E*	*D*	*T*	*H*	*E*	*P*	*U*	*Z*	*Z*	*L*	*E*

10. (a) 4, 6, 24. $\sigma(3) = 1 + 3 = 4$, $\sigma(5) = 1 + 5 = 6$, and $\sigma(15) = 1 + 3 + 5 + 15 = 24$.

    (b) 2047. $\sigma(2^{10}) = 1 + 2 + 2^2 + \ldots + 2^{10}$. In the notation of problem 7, this is $S(10)$. From problem 7c, we know that $S(10) + 1 = 2048$. Therefore $\sigma(2^{10}) = S(10) = 2047$.

    (c) 2048. Since $2^{11} - 1$ is prime, $\sigma(2^{11} - 1) = 1 + 2^{11} - 1 = 2^{11} = 2048$.

    (d) 2047(2048). Because $2^{11} - 1$ is prime and it is greater than 2^{10}, every factor of $2^{10}(2^{11} - 1)$ is a factor of 2^{10} or is a factor of 2^{10} multiplied by $2^{11} - 1$. The sum of all the factors of the first type is 2047 from part b of this problem. The sum of all factors of the second type is therefore $2047(2^{11} - 1)$. Adding these gives $2047(2^{11})$. Note that $\sigma(2^{10}(2^{11} - 1)) = 2[2^{10}(2^{11} - 1)]$. Numbers n with the property $\sigma(n) = 2n$ are called perfect numbers.

11. and 12.

(a) 136080. There are 9 choices for the most significant digit (because 0 is not an option), 9 for the next most significant digit (because 0 is now an option but whatever was selected as the most significant digit cannot be used), 8 for the next, and so on: $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080$.

(b) 168570. Using the argument from the previous step, there are 9 single-digit numbers, 9×9 two-digit numbers, $9 \times 9 \times 8$ three-digits numbers and so on. So there are $9 + (9 \times 9) + (9 \times 9 \times 8) + (9 \times 9 \times 8 \times 7) + (9 \times 9 \times 8 \times 7 \times 6) + (9 \times 9 \times 8 \times 7 \times 6 \times 5)$ numbers with at most six digits. You could compute this keeping a running total, and using partial products as you go along. For example, start with $9 + 81 = 90$. The next term to add is $81 \times 8 = 648$. So the new total is $90 + 648 = 738$. The next term to add is 648×7 and so on. The computation is easier, however, if you recognize that you can factor 9 from every term in the sum

$$9(1 + 9 + (9 \times 8) + (9 \times 8 \times 7) + (9 \times 8 \times 7 \times 6) + (9 \times 8 \times 7 \times 6 \times 5)).$$

One can now factor 9 from all but the first term, and continue this factoring to get:

$$9(1 + 9(1 + 8(1 + 7(1 + 6(1 + 5))))).$$

Evaluate this by starting with the innermost parentheses $(1 + 5)$. Iteratively add 1 and multiply until the computation is complete.

(c) There are 168570 numbers with six or fewer digits. Using the argument from part a, there are $136080 \times 4 = 544320$ seven digit numbers, so there are $168570 + 544320 = 712890$ numbers with seven or fewer digits. The 288657th number therefore must be a seven digit number. The 288657th smallest number is the $288657 - 168570 = 120087$th largest 7-digit number (subtracting the number of numbers with fewer than 7 digits). There are $544320/9 = 60480$ numbers with the leading digit fixed for each of the nine possible leading digits. That is, there are 60480 with leading digit 1, 60480 with leading digit 2 and so on. 120087/60480 has a quotient of 1 and a remainder of 59607, so the leading digit is the second smallest or 2. Therefore the number we seek is the 59607th largest of the 7-digit numbers that start with 2. There are $50480/9 = 6720$ numbers with leading digit 2 and next digit fixed to any one of the nine remaining digits (0,1,3,4,5,6,7,8,9). Dividing the remainder from the first part by 6720 is $59607 = (8)6720 + 5847$. The ninth remaining digit (counting from zero) in this list is 9. So the next digit is 9 and the number we seek is the 5847th smallest of the numbers that start with 29. There are $6720/8 = 840$ seven-digit numbers with the first two digits 29. Continuing in this fashion:

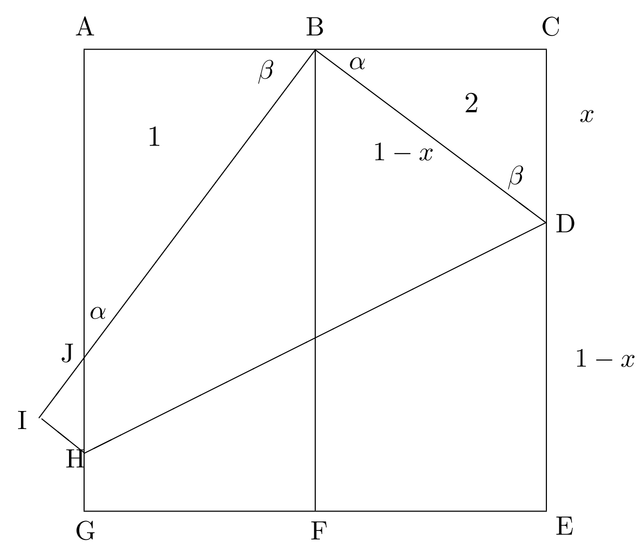
digit significance	remaining digits	index	digit
1	$1, 2, 3, 4, 5, 6, 7, 8, 9$	$120087 = (1)60480 + 59607$	2
2	$0, 1, 3, 4, 5, 6, 7, 8, 9$	$57607 = (8)6720 + 5847$	9
3	$0, 1, 3, 4, 5, 6, 7, 8$	$5847 = (6)840 + 807$	7
4	$0, 1, 3, 4, 5, 6, 8$	$807 = (6)120 + 87$	8
5	$0, 1, 3, 4, 5, 6$	$87 = (4)20 + 7$	5
6	$0, 1, 3, 4, 6$	$7 = (1)4 + 3$	1
7	$0, 1, 3, 4, 6$	3	4

Note that if the quotient is x, then the number is bigger than all the first x groups and the next digit is the $(x+1)$st smallest of those remaining. For example, for the fourth most significant bit, the quotient is 6 so the next digit is the 7th smallest of those remaining, in this case 8. So the 288657-th number is 2978514. Mapping 1 to A, 2 to B, and so on, the check phrase is BIGHEAD.

13. (a) $\frac{3}{8}$. Throughout this solution, we will use a segment name (such as AB) to refer to the segment's length. Because AC is the side of a unit square, $AC = 1$. Because B is the midpoint of AC, we have $BC = \frac{1}{2}$. Let x be the length of CD, then $DE = 1 - x$. By construction, $DE = DB$ (since point E lies at B after the folding; these are equivalent sides of the two congruent quadrilaterals). Therefore $DB = 1 - x$. Triangle BCD is a right triangle, so applying the Pythagorean theorem, simplifying and solving yields the value of CD:

$$\begin{aligned} \left(\frac{1}{2}\right)^2 + x^2 &= (1-x)^2 \\ \frac{1}{4} + x^2 &= 1 - 2x + x^2 \\ \frac{1}{4} &= 1 - 2x \\ 2x &= \frac{3}{4} \\ x &= \frac{3}{8}. \end{aligned}$$

(b) $\frac{1}{3}$. We'll show that $AJ = \frac{2}{3}$. Then, since $AG = 1$, we have $GJ = \frac{1}{3}$. Triangle ABJ (labeled triangle 1 in the figure) is similar to triangle BCD (labeled triangle 2 in the figure). They are both right triangles, since angles JAB and BCD are right angles. Angle ABJ is complementary to angle CBD (they sum to 90 degrees), since angle JBD is a right angle (corresponding to angle FED after the fold) and the three create a straight line (180 degrees). However, angle ABJ is also complementary to angle AJB, since they are the non-right angles in a right triangle. Therefore, angle AJB is the same



as angle CBD. Triangle 2 is a 3-4-5 triangle, with the "3" side opposite angle DBC (of size α in the figure). Since $AB = 1/2$, we use the proportionality of sides in similar triangles:

$$\begin{aligned} \frac{\frac{1}{2}}{AJ} &= \frac{3}{4} \\ 2 &= 3AJ \\ AJ &= 2/3. \end{aligned}$$