Solutions for Go Figure 2002

1. 977.5. 15% of 1000 is $1000 \times .15 = 150$, so $B = 1150$. C is 85% of B. $1150 \times .85 = 977.5$. Notice that this does not take us back to where we began at 1000. 15% of 1000 is less than 15% of 1150, so we will subtract more in the second step than we add in the first.

2. 28%. There are $.3 \times 600 = 180$ pennies in the jar and $.2 \times 150 = 30$ pennies in the coffee can. Once the two groups of coins are combined, there are $180 + 30 = 210$ pennies and $600 + 150 = 750$ total coins. Therefore the fraction of pennies is $\frac{210}{750} = .28$. This is $\frac{28}{100}$ or 28%. Note that the final percentage of pennies in the bucket is not the arithmetic mean (average) of the original percentages. This is because there were more total coins in the jar and therefore it more greatly effected the final percentage of coins.

3. $\frac{12}{13}, \frac{84}{97}, \frac{7}{9}, \frac{42}{55}, \frac{21}{31}$. There are many ways to solve this problem. You can cross multiply each pair of fractions (10 comparisons) to put all in order. You can guess an order, for example, by guessing that $\frac{12}{13} > \frac{21}{31}$, since the second fraction is probably close to $\frac{2}{3}$, and then use cross multiplication among the adjacent pairs to check the order. This would only be four comparisons if you are a good guesser. Alternatively, you can find a common denominator. However, in this case, the easiest strategy is to find a common numerator. Choosing a least common numerator of 84, and maintaining the original order, the five fractions, become: $\frac{84}{108}, \frac{84}{91}, \frac{84}{124}, \frac{84}{110}, \frac{84}{97}$. When numerators are all equal, the largest fraction has the smallest denominator. Putting the fractions in order from smallest to largest denominator gives the stated ordering.

4. (a) 309. In this progression, the terms are $3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4 \ldots$ The 103rd term is $3 \times 103 = 309$.

   (b) 88. The sequence is an arithmetic progression with a fixed difference of 5. If this sequence had begun with 5, we would only have to divide the final term by 5 to find the number of terms. But we can compare sequences with the same fixed difference and length. Therefore, the sequence $5, 10, 15, 20 \ldots, 440$ (where each term is 1 less that the corresponding term in our sequence) has the the same number of terms as our sequence. So the number of terms is $440/5 = 88$.

   A "classic" method to determine the number of sequence elements between a start value and an end value is to compute the span of the sequence (difference between the first and last term): $441 - 6 = 435$. Dividing by the difference between successive terms (in this case 5) gives us the number of terms *after the first one*: $435/5 = 87$ terms after the first one. We must then add one for the first term, so there are 88 in total. You should use this method with caution since it is easy to forget to add one for the first term.

   (c) $6 + 441 = 447$.

   (d) $11 + 436 = 447$.

   (e) 19668. If one continues the pattern shown in parts c and d, adding the terms of the sequence in pairs moving forward from the beginning and backward from the end (i.e. pairing terms i from the beginning and i from the end), always gives a sum of 447. This is because the increase in the earlier term (5) is exactly the decrease in the later term. There are $88/2 = 44$ such pairs, since there are 88 terms in the sequence. Therefore the sum of all the terms is $44 \times 447 = 19668$.

5. $A = 2$, $B = 1$, $C = 7$. We observe that $5A4$ is greater than 500, so if B is at least 2, then BCC is at least 200 and the product would be at least $100,000$. Since our product has only five digits, we know that B is less than 2. If B were 0, then BCC would be less than 100. Since $5A4$ is less than 600, the product would be at most 60000 which is too small (the product is greater than 90000). Therefore $B = 1$. Now consider the units digit of the product. This is the units digit of the product $4 \times C$. There are only two single-digit numbers that when multiplied by 4 have a units digit of 8. They are 2 and 7. If $C = 2$, then the product is no more than $600 \times 122 = 73200$. This is too small, so $C = 7$. At this point, there are at most four choices remaining for A, and we could try them all. However, we can also determine the value of A more directly by considering the tens digit of the product. The product may be expressed as $(4 \times 177) + (A \times 10 \times 177) + (5 \times 100 \times 177)$. The first term $4 \times 177 = 708$, so it contributes nothing to the tens digit of the product. The last term, because it's multiplied by 100, also contributes nothing to the tens digit of the product. Therefore, the tens digit of the product is the units digit of $A \times 7$. There is only one product of a single digit times 7 with a units digit of 4, namely 2 ($7 \times 2 = 14$). Therefore $A = 2$. Doing a final check $524 \times 177 = 92748$.

6. (a) One possible solution is 1..10. As long as the interval of size 10 starts with a number that has a remainder of $1, 2, 3$ or 4 when divided by 7, then the interval will have only one number divisible by 7.

   (b) One possible solution is 5..14. As long as the interval starts with a number that is divisible by 7, or has a remainder of 5 or 6, then there will be two numbers divisible by 7 in an interval of size 10.

   (c) 432. 2028 has a remainder of 5 when divided by 7. Therefore the first number in the interval that is divisible by 7 is 2030 (a number two greater will have a remainder of zero when divided by 7). 5048 has a remainder of 1 when divided by 7, so the last number in the interval that is divisible by 7 is 5047. The numbers in the interval that are divisible by 7 now form an arithmetic progression $2030, 2037, \ldots, 5047$. We can determine the number of terms in this progression by either of the methods described in problem 4. For example, using the span method, the difference between the first and last elements is $5047 - 2030 = 3017$. $3017/7 = 431$ is the number of terms after the first. Adding one for the first term yields a total of 432 terms, numbers divisible by 7 in the interval. One doesn't have to think directly in terms of arithmetic progressions to solve this problem. Once you have determined the first and last terms divisible by 7, you can observe that every 7th term is divisibly by 7 and compute the number using the span method, for example.

7. (a) $243 = 3^5$ with a 3-score of 5. No other number in this interval is divisible by 243.

   (b) 3. There are three numbers in the interval divisible by $3^4 = 81$. They are $81, 162$, and 243. The interval 1..243 has $243 = 3 \times 81$ terms, so there are three elements divisible by 81, and there are no additional terms divisible by 81 in the interval 244..250.

   (c) 123. We count the total 3-score by counting all the numbers with 3-score 5, all the numbers with 3-score at least 4, all the numbers with 3-score at least 3, and so on. A number with 3-score k will be counted k times as it should be. We start by counting only the interval 1..243. There is one number with 3-score 5, and 3 numbers with 3-score at least 4. Using the same argument as we used in part (b), there are $3 \times 3 = 9$ numbers with 3-score at least 3 (three between each consecutive pair of numbers with 3-score at least 4), there are $9 \times 3 = 27$ numbers with 3-score at least 2, and there are $27 \times 3 = 81$ numbers with a 3-score at least 1. Thus there is a total 3-score of $1+3+9+27+81 = 121$ in the interval 1..243. In the interval 244..250, there are only two numbers divisible by 3, namely 246 and 249 and neither of these is divisible by 9. Therefore this end interval contributes 2 to the total 3-score, giving a total 3-score of $121 + 2 = 123$.

(d) 123. The interval 244..493. Is the interval 1..250 moved up by 243. We have to determine the 3-score of a number of the form $3^5 + x$ for an x in the interval 1..250. Notice that 493 is not big enough to have a 3-score of 6 ($3^6 = 729$). The number $3^5 + x$ is divisible by 3 if and only if x is divisible by 3. The same argument holds for $9, 27, 81,$ and 243. That is, the 3-score of $3^5 + x$ is exactly the 3-score of x as long as x is small enough that the resulting number $3^5 + x$ cannot have a 3-score of 6. Therefore the sum of the 3-scores of the interval 244..493 is exactly the sum of the 3-scores of interval 1..250 computed in part c.

8. (a) 6. We must decide the number of twos in the multiset. Once we have chosen the number of twos, then we are forced to pick exactly enough ones to create a sum of 10. Since $10 = 2 \times 5$, we can pick anywhere from 0 to 5 twos for a total of 6 choices.

(b) 12. Once we decide how many sixes to place in the multiset (zero, one, or two), then we are left with a problem similar to part (a), namely, how many ways to use $\{1, 2\}$ to create a total sum (with the sixes) of 12. If we use 2 sixes, then there is 0 left to be filled out with ones and twos. Therefore there is 1 such set. If we choose one six, then we must choose a multiset with sum $12 - 6 = 6$ from the remaining base of $\{1, 2\}$. Using an argument similar to part (a), we see there are 4 ways to do this (choose $0, 1, 2,$ or 3 twos). Finally if we choose no sixes, then there are 7 ways to get a sum of 12 using ones and twos. Therefore the total number of sets is $1 + 4 + 7 = 12$.

(c) 1499500. $5994 = 6 \times 999$, so there are 1000 choices for the number of sixes starting with 999, decreasing to 0, each with an increasing number of ways to fill out the difference with ones and twos. The pattern shown in part (b) continues. Each time we decrease the number of sixes by one, we have 3 more choices for filling out the balance. Therefore, we much compute the sum of the arithmetic progression $1, 4, 7, \ldots, 2998$. We can sum using the methods described in problem 4. The sum of the first and last term is 2999. We can create 500 such pairs by choosing the smallest and largest remaining numbers iteratively. Therefore the sum $1 + 4 + 7 + \ldots + 2998 = 2999 \times 500 = 1499500$.

9. (a) 8. For each of the three flower types, we must choose whether to plant it or not (2 options). These are independent choices (all can be combined), so there are $2 \times 2 \times 2 = 2^3 = 8$ types of gardens.

(b) 32. This is the same as the previous question except that we now must make 5 independent choices, each with 2 options. So the number of different gardens is $2^5 = 32$.

(c) 11. From part (a), there are 8 gardens that add only flowers (or nothing). By a similar argument, there are $2 \times 2 = 4$ gardens that add only herbs or nothing. Doing nothing was an option for both the flower garden and the herb garden, but otherwise, the choices are disjoint (nothing else shared). Therefore there are $8 + 4 - 1 = 11$ possible types of garden that do not mix flowers and herbs.

(d) 10. We must choose 3 out of the 5 possibilities. There are a number of ways to solve this problem. It is small enough to enumerate the choices. In this case, it is easier to enumerate the ways to pick the two seed types you do *not* want to plant. To calculate this quantity, you begin by picking the first seed to discard. There are 5 choices. Once you've picked that, pick one of the 4 remaining (there are 4 choices). This gives $5 \times 4 = 20$. However, the order of the eliminations doesn't matter: you get the same garden if you throw out oregano, then cilantro that you'd get if you through out cilantro then oregano). Since we counted each option twice, we divide by 2 to correct, giving a total of $20/2 = 10$.

This general procedure is captured in the "choose" notation. The symbol $\binom{n}{k}$, read "n choose k", stands for the number of distinct subsets with exactly k elements that can be chosen from a set of n elements. By definition $\binom{n}{0} = 1 = \binom{n}{n}$. In general

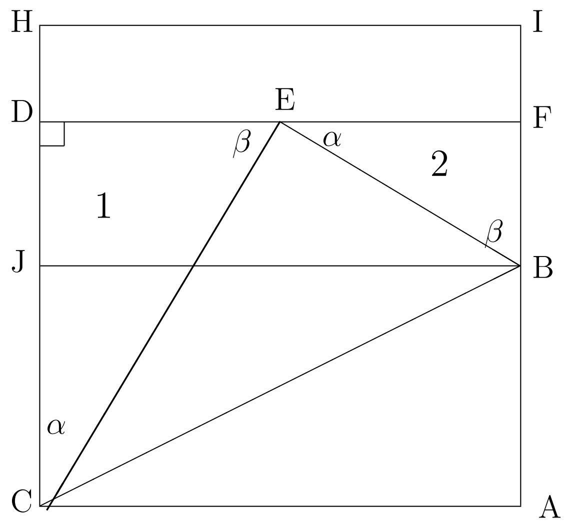
$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots k} \text{ for any } r = 1, 2, \ldots, n-1.]$$

The numerator represents the ordered choice of the k elements. But there are $k!$ ways to order within the set of size k. We divide by this number to correct the overcount.

10. and 11. 499. A number with k trailing zeros can be written as $n \times 10^k$, for some integer n that is not divisible by 10. Consider the prime factorization of 2002!. This is just the product of the prime factorizations of every number from 1 to 2002. For any positive integer $n > 1$, the exponent of 5 in $n!$ is smaller than the exponent for 2 in $n!$. This is because more numbers between 1 and n are divisible by 2 than by 5, more are divisible by $2^2 = 4$ than by $5^2 = 25$, and so on. The prime factorization of 10 is 5×2. Because there are plenty of 2's to match any 5 in the prime factorization of 2002!, the number of trailing zeros is exactly the exponent of 5 in its prime factorization. This is the number of times 5 appears as a (repeated) factor in the interval 1..2002. Define the 5-score of a number the way we defined 3-score in problem 7 (the exponent of 5 in the prime factorization of that number). Then the number of trailing zeros is the sum of the 5-score of the numbers in the interval 1..2002. We can compute this using the same techniques used in problem 7.

Since $5^4 = 625$ and $5^5 = 3125$, the highest 5-score in the interval is 4. We begin by computing the sum of the 5-scores in the interval 1..625. There is one number with 5-score 4, 5 numbers with 5-score at least 3, 25 numbers with 5-score at least 2 and 125 with 5-score at least 1. Thus the sum of the 5-score in 1..625 is $1 + 5 + 25 + 125 = 156$. By an argument similar to that in problem 7(d), the sum of the 5-scores in 626..1250 and 1251..1875 are the same, so the sum of the 5-scores in 1..1875 is $3 \times 156 = 468$. Again, the same argument shows that the sum of the 5-scores in 1876..2002 is the same as the sum of the 5-scores in 1..127. This is $1 + 5 + 25 = 31$, since 125 has the highest 5-score (of 3). Thus the total sum in the interval 1..2002 is $468 + 31 = 499$.

12. 2x. Throughout this solution, we will use a segment name (such as AC) to refer to the segments length. Because it is a side of the square, $AC = 1$. Since AB is half the side of the square, $AB = \frac{1}{2}$. Because trangle ECB is the reflection of triangle ABC, we have $EC = AC = 1$ and $EB = AB = \frac{1}{2}$. Triangle CDE (labeled triangle 1 in the figure) is similar to triangle BEF (labeled triangle 2). They are both right triangles, since angles CDE and



BFE are right angles. Angle FEB is complementary to angle DEC (they sum to 90 degrees), since angle CEB is a right angle and the three add to a straight line (180 degrees). However, angle DEC is also complementary to angle DCE, since they are the non-right angle in a right triangle. Therefore, angle DCE is the same as angle FEB. Thus right triangles CDE and BFE are similar, with the former twice as large. Side BF corresponds to side DE, so DE is twice as large.

There are other valid expressions for the length of DE as a function of x. Any expression is acceptable as long as that expression evaluates to .6 when $x = .3$. For example, by the Pythagorean Theorem on right triangle EFB, we know that $EF^2 + x^2 = (\frac{1}{2})^2$. Solving for EF, we have $EF = \sqrt{\frac{1}{4} - x^2}$. Since $DE + EF = 1$, we have $DE = 1 - \sqrt{\frac{1}{4} - x^2}$. Similarly, one could observe that $CJ = \frac{1}{2}$ and $DJ = x$. Using the Pythagorean Theorem on right triangle DEC, we have $DE^2 + (x + \frac{1}{2})^2 = 1$. Therefore $DE = \sqrt{1 - (x + \frac{1}{2})^2}$. It's still possible to compute the value of x in the next problem using these expressions, but the algebra becomes significantly more complicated.

13. $\frac{3}{10}$. Let y be the length of EF. From the Pythagorean theorem on triangle BEF, we have $x^2 + y^2 = \frac{1}{4}$. Because $DE = 2x$ and $DF = 1$ (the same as the square side), we have $2x + y = 1$. Solving for y in the second equation, we have $y = 1 - 2x$. Substituting into the first equation and simplifying:

$$\begin{aligned}
x^2 + (1-2x)^2 &= \frac{1}{4} && (1)\\
x^2 + 1 - 4x + 4x^2 &= \frac{1}{4} && (2)\\
5x^2 - 4x + \frac{3}{4} &= 0 && (3)\\
20x^2 - 16x + 3 &= 0 && (4)\\
(10x - 3)(2x - 1) &= 0 && (5)
\end{aligned}$$

We cannot have $x = \frac{1}{2}$, since that is the length of the hypotenuse of triangle BEF. This would imply $y = 0$, which, given the figure, we assume is not true. Therefore $x = \frac{3}{10}$. Note that it may have been easier to use the quadratic formula rather than factoring in step 5.