1. (a) 12 (or 6 + 6), (b) 17 (or 8 + 9)

2. The ratio is 2/3 so $a = 64/27$ and $b = 128/81$.

3. {c, d} = {47, 53}. The pattern of the products in the hint can be seen more easily in column format:

$$\begin{aligned} 7 \times 13 &= 91 \\ 17 \times 23 &= 391 \\ 27 \times 33 &= 891 \\ 37 \times 43 &= 1591 \end{aligned}$$

Each factor is ten larger than the one above it. The next product in this pattern is 47 x 53. Computing the product 47 x 53 = 2491, is required. For those more curious about this pattern, we see it was not a coincidence that the next product ended in 91. Each of these products are of the form $(10n - 3)(10n + 3) = 100n^2 - 9$, for every n the product will end in 91 (always the case when one subtracts 9 from any multiple of 100).

4. (a) This arithmetic progression gives the "times table" for 5. That is, it starts with 5 and each successive number is 5 greater than the one before. Therefore the kth term is equal to k x 5. The 70th term is 70 x 5 = 350.
   (b) Each term is this progression is 3 less than the corresponding term is in the progression is part (a). Therefore the 70th term is 350 - 3 = 347.
   (c) Subtracting 9 from each term, we get 5, 10, 15, ..., 985. Since 985 = 5 x 197, there are 197 terms.
   (d) For this progression, the sum of the first and last terms is 14 + 994 = 1008, and the sum of the second term and the next-to-last term is 19 + 989 = 1008. One could continue matching terms one further away from each end. Each pair has the same sum, since the smaller term grows by 5 and the larger term shrinks by 5. The average of each of these pairs is 1008/2 = 504. For an odd number of terms, there is one unmatched term in the middle equal to (in this case) 504. Therefore the average of all the terms is 504.
   (e) The sum is the average of the terms times the number of terms: 504 x 197 = 99288.

5. (a) $(b^4)^q \times b^r = b^{(4q+r)}$. Thus $123 = 4q + r$. Dividing 123 by 4 we have a quotient $q = 30$ and a remainder $r = 3$.
   (b) For any integer multiplication problem, the units digit is determined by the product of the units digits of the factors. For example, when multiplying a number ending with 2 by a number ending in 9, the units digit will be 8 (from the product 18) regardless of the full value of these factors. Consider the powers of 7. If we are only interested in the units digit, we can do the multiplication with only the units digits, and keep only the units digit of each product. Thus $7^1 = 7$, $7^2 = 49$, and we'll only keep the 9. The units digit of 7^3, therefore is 3, since 9 x 7 = 63. Keeping only the 3, the units digit of 7^4 is 1 since 3 x 7 = 21. If we were to continue, we'd see the same pattern: 7, 9, 3, 1, 7, 9, 3, 1,... We just need to determine the value of the 123rd element in this sequence.

From part (a), we know that $123 = 4 \times 30 + 3$. Since 7 raised to any power that is a multiple of 4 will have a units digit of 1, the units digit of 7^{123} will be the same as 7^3 or 3. One could reach the same answer with a little more work by partially doing the multiplication:

$$\begin{aligned} 7^{123} &= (7^4)^{30} \times 7^3 \\ &= 2401^{30} \times 343. \end{aligned}$$

At this point, you must observe that no matter how many factors of 2401 are multiplied together, the units digit of the product will be 1. Therefore the units digit of 7^{123} is 3, the same as the units digit in 343.

6. First observe that $A = 1$ because otherwise $A2B$ will be greater than 200 and the product will be greater than $200 \times 600 = 120000$, a six-digit number. The product in this problem has only 5 digits. Because the units digit is 1, we have $B = 7$ since $3 \times 7 = 21$ is the only product 3 times a single-digit number that has a 1 in the units place. Therefore the first number is 127. Computing the product, $127 \times 683 = 86741$, which is of the correct form with $C = 4$.

7 and 8.

(a) we compute the prime factorization of 792:

$$\begin{aligned} 792 &= 2 \times 396 \\ &= 2 \times 2 \times 198 \\ &= 2 \times 2 \times 2 \times 99 \\ &= 2 \times 2 \times 2 \times 9 \times 11 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \times 11 \end{aligned}$$

We must group these six factors (three 2's, two 3's, and one 11) into three groups. Then f is the product of the first group, g the product of the second group, and h the product of the third. Experience with problem 1 should make us expect that f, g, and h are as close to each other as possible. One possible solution $\{f,g,h\} = \{8,9,11\}$, with $8 + 9 + 11 = 28$. To prove that this is the smallest sum, we see that one of the factors, say f, must be a multiple of 11 and it must be no larger than 28 in any minimum sum. Therefore $f = 11$ or $f = 22$. If $f = 22$, then $g \times h = 2 \times 2 \times 3 \times 3 = 36$. Problem 1 (a) tells us that the minimum for $g + h$ in this case is 12. This would give $f + g + h = 22 + 12 = 34$ which is bigger than 28. Therefore $f = 11$. Now $g \times h = 2 \times 2 \times 2 \times 3 \times 3 = 72$ and Problem 1 (b) shows that the minimum for $g + h$ is $8 + 9 = 17$. Thus the answer is $11 + 17 = 28$.

(b) The factors must be chosen fom [2,3,4,11} since the sum of factors is reduced by replacing a factor 6 by 3×2, replacing a factor 8 by 2×4 or $2 \times 2 \times 2$, and replacing a factor 9 by 3×3, and so on. The replacement (e.g., 2×3 for 6) has the same product (6) but a smaller sum (5 instead of 6). The factorizations with the smallest sum of factors are $2 \times 4 \times 3 \times 3 \times 11$ and $2 \times 2 \times 2 \times 3 \times 3 \times 11$, each with a sum of 23. For the algebraically inclined, one can observe that $a + b \leq ab$ for any $a > 1$ and $b > 1$, and therefore simply realize the smallest sum breaks 792 into its prime factors.

9 (a) Taking square roots of both sides, we get
$$x - 50 = \pm 3$$
$$x = 50 \pm 3$$
$$= 47 \text{ or } 53.$$

(b) Taking square roots of both sides, we have
$$u^2 - 17 = \pm 8$$
$$u^2 = 17 \pm 8$$
$$u^2 = 9 \text{ or } 25.$$

The four answers are 3, -3, 5, -5.

10. By the Pythagorean theorem, the length y of KL satisfies
$$y^2 + 1122 = 130^2$$
$$y^2 = 130^2 - 112^2$$
$$= (130 - 112)(130 + 112)$$
$$= 18 \times 242$$
$$= 36 \times 121$$
$$= (6 \times 11)^2$$

since y is positive, we have y = 6 x 11 = 66. Next we note that DHJI and DHLK are similar triangles since they share angle KHL and each has a right angle. Let the length of KI be t.

The ratio between the sides of the triangles is $\frac{112 + 56}{112} = \frac{3 \times 56}{2 \times 56} = \frac{3}{2}$. The hypotenuses will have the same ratio so $\frac{HI}{HK} = \frac{t + 130}{130} = \frac{3}{2}$
$$2(t + 130) = 3 \times 130$$
$$2t = 3 \times 130 - 2 \times 130$$
$$2t = 130$$
$$t = 65$$

11. (a) ΔABC is simiilar to ΔDEF since the lengths of DF and EF are double the lengths of AC and CB respectively and the angle between these pairs of proportional sides is a a right angle in each case (and therefore equal). It follows that angle EDF is equal to angle BAC. Now ΔDBG is similar to ΔABC since corresponding angles are equal. The lengths of the hypotenuses DB and AB are 4 and 13 respectivley. The area of ΔDBG is $(4/13)^2$ x (area of ΔABC). The area of ΔABC is 1/2 the base times the height or

$\frac{1}{2} \times 5 \times 12 = 30$. Therefore the area of $\Delta DBG = \frac{16}{169} \times 30 = \frac{480}{169}$.

One would perform the same computation to calculate the area of ΔDBG more "directly" by computing the base and height of triangle DBG ans 4/13 of the corresponding sides of ΔABC.

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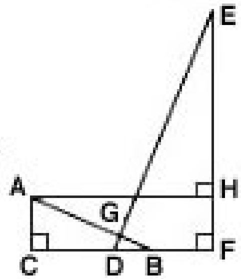
11. (b) Let H be the foot of the perpendicular from A to EF as shown in the figure. Then the lengths of AH and EH are $8 + 4 + 6 = 18$ and $24 - 5 = 19$ respectively. By the Pythagorean theorem, the length of AE is

$$\sqrt{18^2 + 19^2} = \sqrt{324 + 361} = \sqrt{685}.$$



Note: If you didn't see the "trick" of adding segment AH, there is still enough information to compute AE, but the computation is more complicated which will increase our chances of making a computational error. One can work directly with ΔAGE. Angle AGE is a right angle because angle DGB is a right angle (one complementary/supplementary-angle arguments, or see that these are opposite angles from a pair of intersecting lines). ED = 26 by the Pythagorean theorem (on ΔEDF), and

$DG = \frac{4}{13} \times 5$ by similar triangles (from part a). Therefore $EG = 26 - \frac{20}{13}$.

Similarly $AB = 13$, $GB = \frac{4}{13} \times 12$, so $AG = 13 - \frac{48}{13}$. We can then use the Pythagorean theorem on ΔAGE:

$AE^2 = (26 - \frac{20}{13})^2 + (13 - \frac{48}{13})^2$. Finishing the computation, we get the same answer as above.

12. In this case it is easier to count the number of ways in which 2 or 3 girls are *not* assigned to any car, and subtract that from the total number of ways to fill the cars. If no car has 2 or 3 girls, then each of the three cars must have exactly one girl. There are $3 \times 2 = 6$ ways to assign the three girls to the three cars since there are three ways to pick a girl for the first car and after that choice, two ways to pick a girl for the second car, with the remaining girl going in the last car. We must now compute the number of ways to assign the boys to the cars. However, this is the same problem solved in the problem statement except that now there are 5 boys to assign rather than 8 children, and each car can now hold only 2 boys instead of 3 children. The number of ways to assign boys to the cars is:

$$3 \cdot \binom{5}{1} \cdot \binom{4}{2} \cdot \binom{2}{2} = 15 \left(\frac{4 \cdot 3}{2} \right) \cdot 1 = 90.$$

Therefore, there are $6 \times 90 = 540$ ways to not assign 2 or 3 girls to a car. Thus the number of ways to assign 2 or 3 girls to a car is $1680 - 540 = 1140$.