TIME AND COMPLEX HYPERBOLIC TRIGONOMETRY

RUSSELL CLARK ESKEW

Abstract. Let us present the subjects of hyperbolic geometry and complex numbers using hyperbolas, and compare this geodesic model to an Euler's formula complex numbers model seen in a Poincaré disk or a Mobius band. Using that frequency, a transformation of accelerations is made on harmonic motion by taking the the derivatives of a cyclic periodic wave velocity. Then a simple representation for the non-Euclidean angle of parallelism can be applied to the astronomical parallax within a hypothesis of the near-universe space and time. We make the electromagnetic spectrum consistent with variant, rather than constant, radian/second wave velocities on both astronomic and quantum scales. The relativistic Lorentz transformation is changed for this geometry. Distance and time are expressed algebraically.

1. Horizontal, Vertical, Diagonal Hyperbolas and the Circle

Trigonometry originates with Hippocrates (470–410 B.C.) of Chios, Euclid's Elements (325–265 B.C.), Archimedes (287–212 B.C.), Heron (60 A.D.), Ptolemy's Almagest (178) all of Alexandria, and Al Battani (Albatenius) (877-918) of Iraq, among others. Girolamo Cardano (1501–1576), J. Wallis (1685), L. Euler (1707–1783), Caspar Wessel (1797), and J. R. Argand (1806) introduced complex numbers, motivated by solving polynomials with equations such as

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta = x + iy \tag{1}$$

and the *modulus*, or length of z

$$|z| = \sqrt{x^2 + y^2}$$

[1] [18] [20]. In the 19th century, the trigonometry required for measuring a circle potentially changed with the horizontal hyperbola of Christof Gudermann (1798–1852) and the vertical hyperbola of Nikolai Ivanovic Lobacevskii (1792–1856) [7, pp. 376–377] [16, pp. 11–45] and Janos Bolyai (1802–1860) [4].

The subject of Euclidean trigonometry prefers to use $x = r\cos\theta$ and $y = r\sin\theta$. However, this position is not compatible with hyperbolic trigonometry. Figure 1 shows the potential advantages of using hyperbolic trigonometry. Figure 1 (top) illustrates the horizontal hyperbola of Christof Gudermann, $x^2 - y^2 = 1$ [6, pp. 312–313] [3, p. 269]

$$\begin{aligned}
&x = \sec\psi = \cosh\sinh^{-1} y = \coth\alpha && y = \tan\psi = \sinh\sinh^{-1} y = \operatorname{csch}\alpha \\
&\frac{1}{x} = \cos\psi = \operatorname{sech}\sinh^{-1} y = \tanh\alpha && \frac{1}{y} = \cot\psi = \operatorname{csch}\sinh^{-1} y = \sinh\alpha \\
&\frac{x}{y} = \csc\psi = \coth\sinh^{-1} y = \cosh\alpha && \frac{y}{x} = \sin\psi = \tanh\sinh^{-1} y = \operatorname{sech}\alpha.
\end{aligned} \tag{2}$$

1991 *Mathematics Subject Classification.* Primary: 30A99, 51M10, 51P05; Secondary: 83F05.
Key words and phrases. angle of parallelism, complex, de Moivre, electromagnetic spectrum, Euler, frequency, geodesics, parsec, quantum mechanics, relativity, space, time, trigonometry.

The angle ψ is called the *gudermannian* and defined for $0 \leq \psi \leq \frac{\pi}{2}$. When the frequency f *cycles/second* $= y + \sqrt{y^2+1}$, for example, rational numbers $\operatorname{sech}\ln f = \operatorname{sech}\sinh^{-1} y = \tanh\alpha$ are made from $\alpha = \ln\frac{f+1}{f-1}$, therefore $\psi = \cos^{-1}\operatorname{sech}\ln f = \cos^{-1}\tanh\alpha$. Time $T = 1/f$ is the fraction of a second per cycle. All of the equations herein are geodesics.

Figure 1 (bottom) illustrates the vertical hyperbola of Nikolai Ivanovic Lobacevskii, $y^2 - x^2 = 1$ [16, p. 41] [19, pp. 414, 421–423, 434] [7, pp. 376–377],

$$\begin{aligned}
x &= \cot\theta = \sinh\sinh^{-1} x = \operatorname{csch}\alpha & y &= \csc\theta = \cosh\sinh^{-1} x = \coth\alpha \\
\frac{1}{x} &= \tan\theta = \operatorname{csch}\sinh^{-1} x = \sinh\alpha & \frac{1}{y} &= \sin\theta = \operatorname{sech}\sinh^{-1} x = \tanh\alpha \\
\frac{x}{y} &= \cos\theta = \tanh\sinh^{-1} x = \operatorname{sech}\alpha & \frac{y}{x} &= \sec\theta = \coth\sinh^{-1} x = \cosh\alpha.
\end{aligned} \tag{3}$$

We relate angles $\psi = \frac{\pi}{2} - \theta$ since $\cot\psi = \sinh\alpha = \tan\theta = \cot(\frac{\pi}{2} - \theta)$. The set of equation (3) uses frequency

$$\begin{aligned}
e^{\sinh^{-1} x} &= f = x + \sqrt{x^2+1} = \cot\theta + \sqrt{(\cot\theta)^2 + 1} = \cot\theta + \csc\theta \\
&= \sinh\ln f + \cosh\ln f = x + y
\end{aligned}$$

with $x \geq 0$ and $\sinh^{-1} x = \ln|x + \sqrt{x^2+1}| = -\int \csc\theta d\theta$. The derivative $\frac{d}{dx}\sinh^{-1} u = \frac{u'}{\sqrt{u^2+1}} = \frac{1}{\sqrt{x^2+1}} = \frac{1}{\csc\theta} = \sin\theta$ is so when $u = x$ and $\frac{d}{dx}x = 1$.

In fact, hyperbolic trigonometry replaces the Euclidean Euler equation (1), where $z = a + ib$, $\bar{z} = a - ib$, and $|z| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}}$, with

(4)

$$\begin{aligned}
z = r(\sec\psi + i\tan\psi) &= r(x+iy) & |z| &= r\sqrt{(\sec\psi)^2 + (\tan\psi)^2} = r\sqrt{x^2+y^2} \\
z = re^{i\psi} = r(\cos\psi + i\sin\psi) &= r\left(\frac{1}{x} + i\frac{y}{x}\right) & |z| &= r\sqrt{(\cos\psi)^2 + (\sin\psi)^2} = r\sqrt{\left(\frac{1}{x}\right)^2 + \left(\frac{y}{x}\right)^2} = r1 \\
& & &= r\sqrt{(\sec\psi)^2 - (\tan\psi)^2} = r\sqrt{x^2 - y^2} = r1 \\
z = r(\cot\theta + i\csc\theta) &= r(x+iy) & |z| &= r\sqrt{(\csc\theta)^2 + (\cot\theta)^2} = r\sqrt{y^2+x^2} \\
z = re^{i\theta} = r(\cos\theta + i\sin\theta) &= r\left(\frac{x}{y} + i\frac{1}{y}\right) & |z| &= r\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = r\sqrt{\left(\frac{x}{y}\right)^2 + \left(\frac{1}{y}\right)^2} = r1 \\
& & &= r\sqrt{(\csc\theta)^2 - (\cot\theta)^2} = r\sqrt{y^2 - x^2} = r1.
\end{aligned}$$

Since $a = r\cot\theta = rx$ and $b = r\csc\theta = ry$, we thus have the *primary complex number*, $a + bi = r\cot\theta + (r\csc\theta)i = r(\cot\theta + i\csc\theta) = r(x+iy)$. The length of the vector $z = (a,b) = a + ib$ is denoted $|z| = \sqrt{z\bar{z}} = \sqrt{(a+ib)(a-ib)} = \sqrt{a^2+b^2} = r\sqrt{x^2+y^2} = r\sqrt{y^2+x^2} = r\sqrt{(\csc\theta)^2 + (\cot\theta)^2}$. r and $|z|$ are different. The vector makes an angle $\vartheta = \tan^{-1}\frac{b}{a}$, not $\theta = \cos^{-1}\frac{x}{y}$, with the x-axis. Example: $x = 4$, therefore $re^{i\theta} = r(\frac{4}{\sqrt{17}} + i\frac{1}{\sqrt{17}})$, $\quad |z| = r\sqrt{(4 + i\sqrt{17})(4 - i\sqrt{17})} = r\sqrt{33}$.

A similar treatment can be made for the diagonal hyperbola, $2xy = 1$, with coordinate angles $x = \frac{1}{f\sqrt{2}}$ and $y = \frac{f\sqrt{2}}{2}$, where $-2(\cosh\ln f)(\sinh\ln f) = -\sinh 2\ln f = x^2 - y^2$.

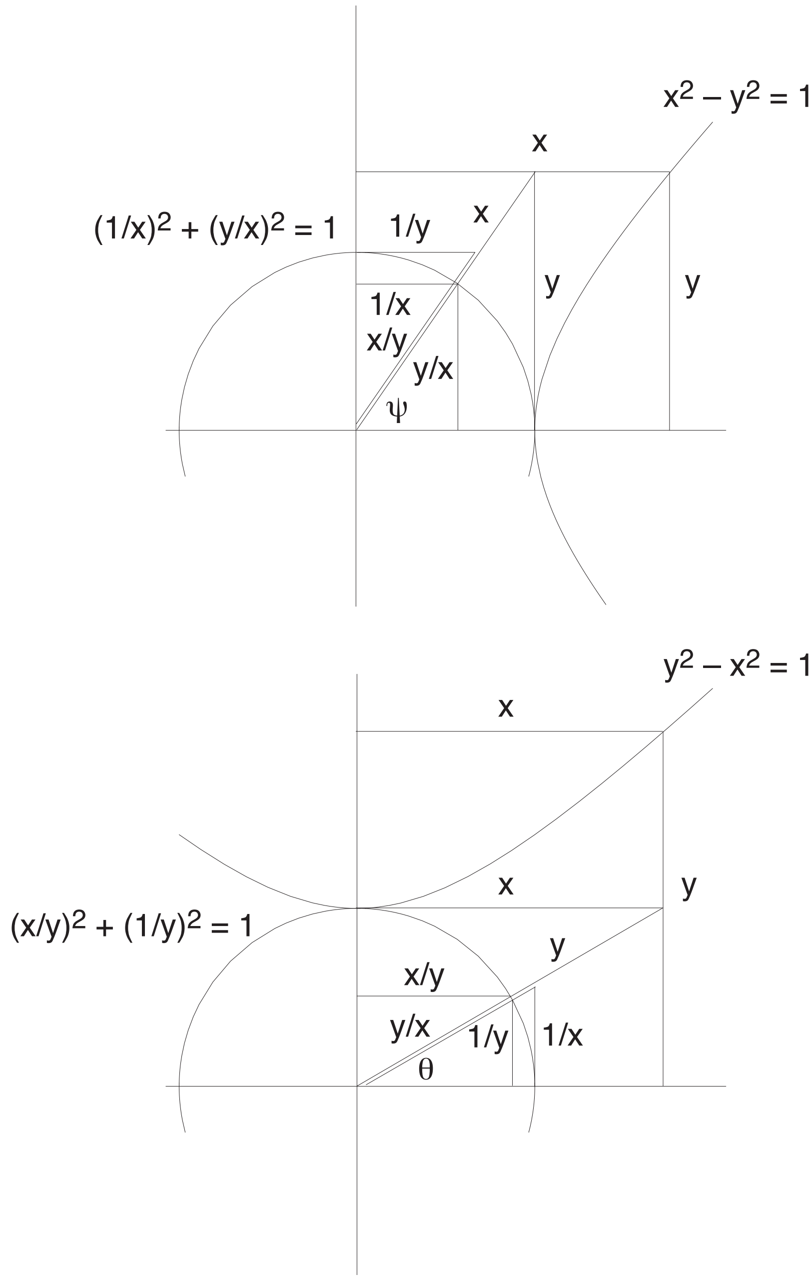

FIGURE 1. Horizontal(top), vertical(bottom) hyperbola and circle coordinate angles as defined in hyperbolic trigonometry

;

2. Roots, products, moduli, de Moivre's, exponential and logarithmic functions

This section is introductory material typical of complex numbers, yet modified for hyperbolic trigonometry. One use of complex numbers is to take square roots of negative real numbers [18, pp. 6–7, 12–13, 27–34, 41–42] [1, pp. 3–13]. First, we wish to prove this by showing that $u^2 = z$, where $z = a + ib$ and $u = r(x + iy)$ are both complex numbers. Then

$$\begin{aligned}
u^2 &= a + ib \\
&= [r(x+iy)][r(x+iy)] = r^2[(x^2 - y^2) + i2xy] \\
&= r^2[((\cosh \ln f)^2 - (\sinh \ln f)^2) + i2(\cosh \ln f)(\sinh \ln f)] \\
&= r^2[((\sec \psi)^2 - (\tan \psi)^2) + i2(\sec \psi)(\tan \psi)] \\
&= r^2\left[1 + i2\left(\frac{f^2+1}{2f}\right)\left(\frac{f^2-1}{2f}\right)\right] = r^2\left[1 - i\left[\left(\frac{1}{f\sqrt{2}}\right)^2 - \left(\frac{f\sqrt{2}}{2}\right)^2\right]\right] \\
&= r^2\left[1 + i\frac{f^2 - f^{-2}}{2}\right] = r^2[1 + i \sinh 2 \ln f]
\end{aligned}$$

simultaneously solves the horizontal and diagonal hyperbolas. Hence $a = r$ and $b = r \sinh 2 \ln f = r2\left(\frac{f^2+1}{2f}\right)\left(\frac{f^2-1}{2f}\right) = -r\left[\left(\frac{1}{f\sqrt{2}}\right)^2 - \left(\frac{f\sqrt{2}}{2}\right)^2\right] = r\frac{f^2-f^{-2}}{2}$. Here we find a complementary double-angle formula $[r(x+iy)]^2 = [r(\cot\theta + i\csc\theta)]^2 = [r(\sinh \ln f + i \cosh \ln f)]^2 = r^2(-1 + i \sinh 2 \ln f)$.

We see the double-angle when we consider $z = a + ib$ and $v = r(\cos\psi + i \sin\psi)$ for $v^2 = z$

$$\begin{aligned}
v^2 &= a + ib \\
&= [r(\cos\psi + i\sin\psi)][(r(\cos\psi + i\sin\psi)] \\
&= r^2[[(\cos\psi)^2 - (\sin\psi)^2] + [i2(\cos\psi)(\sin\psi)]] \\
&= r^2[[(\operatorname{sech} \ln f)^2 - (\tanh \ln f)^2] + [i2(\operatorname{sech} \ln f)(\tanh \ln f)]] \\
&= r^2\left[\left[\left(\frac{1}{x}\right)^2 - \left(\frac{y}{x}\right)^2\right] + \left[i2\left(\frac{1}{x}\right)\left(\frac{y}{x}\right)\right]\right] \\
&= r^2\left[\left[\left(\frac{2f}{f^2+1}\right)^2 - \left(\frac{f^2-1}{f^2+1}\right)^2\right] + \left[i2\left(\frac{2f}{f^2+1}\right)\left(\frac{f^2-1}{f^2+1}\right)\right]\right] \\
&= r^2[\cos 2\psi + i \sin 2\psi].
\end{aligned}$$

For we now compute

(5)

$$
\begin{aligned}
r^2[1]^2 = r^2\left[\left(\frac{1}{x}\right)^2+\left(\frac{y}{x}\right)^2\right]^2 &= r^2\left[\left[\left(\frac{1}{x}\right)^2-\left(\frac{y}{x}\right)^2\right]^2+\left[2\left(\frac{1}{x}\right)\left(\frac{y}{x}\right)\right]^2\right] = r^2(x^2-y^2)^2\\
r^2[(\cos\psi)^2+(\sin\psi)^2]^2 &= r^2[[(\cos\psi)^2-(\sin\psi)^2]^2+[2(\cos\psi)(\sin\psi)]^2]\\
r^2[(\operatorname{sech}\ln f)^2+(\tanh\ln f)^2]^2 &= r^2[[(\operatorname{sech}\ln f)^2-(\tanh\ln f)^2]^2+[2(\operatorname{sech}\ln f)(\tanh\ln f)]^2]\\
&= r^2\left[\left[\left(\frac{2f}{f^2+1}\right)^2-\left(\frac{f^2-1}{f^2+1}\right)^2\right]^2+\left[2\left(\frac{2f}{\rho^2+1}\right)\left(\frac{f^2-1}{f^2+1}\right)\right]^2\right]\\
&= r^2[[\cos 2\psi]^2+[\sin 2\psi]^2]\\
&= a^2+b^2.
\end{aligned}
$$

Hence $r\left[\left(\frac{1}{x}\right)^2+\left(\frac{y}{x}\right)^2\right] = \sqrt{a^2+b^2}$, with $a = r\cos 2\psi = r\left[\left(\frac{2f}{f^2+1}\right)^2-\left(\frac{f^2-1}{f^2+1}\right)^2\right]$ and $b = r\sin 2\psi = r\left[2\left(\frac{2f}{f^2+1}\right)\left(\frac{f^2-1}{f^2+1}\right)\right]$. So $r(\frac{1}{x})^2 = (a+\sqrt{a^2+b^2})/2$ and $r(\frac{y}{x})^2 = (-a+\sqrt{a^2+b^2})/2$. If we let $\sqrt{(\frac{1}{x})^2} = \sqrt{(a+\sqrt{a^2+b^2})/2r}$ and $\sqrt{(\frac{y}{x})^2}$ $= \sqrt{(-a+\sqrt{a^2+b^2})/2r}$, we conclude that the equation $v^2 = z$ has the solution $r\sqrt{(\frac{1}{x})^2}+r\sqrt{(\frac{y}{x})^2}i$. The quadratic equation $az^2+bz+c=0$ for complex numbers a, b, c has solutions $z = (-b^{\pm}\sqrt{b^2-4ac})/2a$, where $\sqrt{w}$ is the square root of complex number w. Also hence $r(x^2-y^2) = \sqrt{a^2+b^2}$. We can now expand the modulus squared $|z|^2 = r^2(x^2-y^2) = r^2(1) = r^2(x+y)(x-y) = a^2+b^2 = (a+ib)(a-ib) = z\bar{z}$.

When the polar coordinate representation is $z = r\cos\psi + ir\sin\psi = r\frac{1}{x}+ir\frac{y}{x} = a+ib$, we have the polar modulus $|z| = r\sqrt{(\frac{1}{x})^2+(\frac{y}{x})^2} = r\sqrt{x^2-y^2} = |r\frac{1}{x}+ir\frac{y}{x}| = |a+ib| = \sqrt{a^2+b^2}$ [1, pp. 12–13]. When the hyperbolic form is $z = r(\sec\psi + i\tan\psi) = r(x+iy) = a+ib$ we have the *hyperbolic modulus* $|z| = r\sqrt{x^2+y^2} = r\sqrt{(\sec\psi)^2+(\tan\psi)^2} = |rx+iry| = |a+ib| = \sqrt{a^2+b^2}$ as in equations (5) and (4). This differs from the conventional $|z| = \sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}$ [20].

We calculate the product $z^2 = [r(\cos\theta + i\sin\theta)]^2 = [r(\tanh\ln f + i\,\mathrm{sech}\ln f)]^2 = r^2(((\cos\theta)(\cos\theta)-(\sin\theta)(\sin\theta))+i((\cos\theta)(\sin\theta)+(\cos\theta)(\sin\theta))) = r^2(((\tanh\ln f)^2-(\mathrm{sech}\ln f)^2)+i2(\tanh\ln f)(\mathrm{sech}\ln f)) = r^2(\cos 2\theta + i\sin 2\theta)$ therefore we know the products and roots of de Moivre's formula $z^n = r^n(\cos n\theta + i\sin n\theta)$. The hyperbolic analogue is [1, pp. 15–16]

$$\begin{aligned}
z^n &= r^n(\cos n\psi + i\sin n\psi) & &= r(\cos\varkappa + i\sin\varkappa)\\
&= r^n(\cos(\varkappa + k2\pi) + i\sin(\varkappa + k2\pi)) & &= r^n(\mathrm{sech}\ln f + i\tanh\ln f)^n\\
z_k &= r^{1/n}(\cos(\frac{\varkappa}{n} + \frac{k2\pi}{n}) + i\sin(\frac{\varkappa}{n} + \frac{k2\pi}{n}))\\
&= r^{1/n}(\cos\psi + i\sin\psi) & &= r^{1/n}(\mathrm{sech}\ln f + i\tanh\ln f)\\
z^n &= r^n(\cos n\theta + i\sin n\theta) & &= r(\cos\varpi + i\sin\varpi)\\
&= r^n(\cos(\varpi + k2\pi) + i\sin(\varpi + k2\pi)) & &= r^n(\tanh\ln f + i\,\mathrm{sech}\ln f)^n\\
z_k &= r^{1/n}(\cos(\frac{\varpi}{n} + \frac{k2\pi}{n}) + i\sin(\frac{\varpi}{n} + \frac{k2\pi}{n}))\\
&= r^{1/n}(\cos\theta + i\sin\theta) & &= r^{1/n}(\tanh\ln f + i\,\mathrm{sech}\ln f)
\end{aligned}$$

$k = 0, 1, \ldots, n-1$, upon $\psi = \cos^{-1}\mathrm{sech}\ln f$ or $\theta = \sin^{-1}\mathrm{sech}\ln f = \frac{\pi}{2} - \psi$.

Also we know $z^n = [r(x+iy)]^n = [r(\cot\theta + i\csc\theta)]^n = [r(\sinh\ln f + i\cosh\ln f)]^n = r^n(-1 + i\sinh n\ln f)$. So we have an alternate kind of double-angle formula

$$\begin{aligned}
z^n &= r^n(\sec\psi + i\tan\psi)^n & &= r(\sec\varkappa + i\tan\varkappa)\\
&= r^n(\sec(\varkappa + k2\pi) + i\tan(\varkappa + k2\pi)) & &= r^n(1 + i\sinh n\ln f)\\
z_k &= r^{1/n}(\sec(\frac{\varkappa}{n} + \frac{k2\pi}{n}) + i\tan(\frac{\varkappa}{n} + \frac{k2\pi}{n}))\\
&= r^{1/n}(\sec\psi + i\tan\psi) & &= r^{1/n}(1 + i\sinh n\ln f)\\
z^n &= r^n(\cot\theta + i\csc\theta)^n & &= r(\cot\varpi + i\csc\varpi)\\
&= r^n(\cot(\varpi + k2\pi) + i\csc(\varpi + k2\pi)) & &= r^n(-1 + i\sinh n\ln f)\\
z_k &= r^{1/n}(\cot(\frac{\varpi}{n} + \frac{k2\pi}{n}) + i\csc(\frac{\varpi}{n} + \frac{k2\pi}{n}))\\
&= r^{1/n}(\cot\theta + i\csc\theta) & &= r^{1/n}(-1 + i\sinh n\ln f).
\end{aligned}$$

The exponential and logarithmic functions are

$$\begin{aligned}
e^z &= e^{rx}e^{iry}\\
&= e^{r\cot\theta}e^{ir\csc\theta} = e^{r\cot\theta}(\cos r\csc\theta + i\sin r\csc\theta)\\
&= e^{r\sinh\ln f}e^{ir\cosh\ln f} = e^{r\sinh\ln f}(\cos r\cosh\ln f + i\sin r\cosh\ln f)\\
&= |e^z|\frac{e^z}{|e^z|},\\
\ln z &= \ln|z| + iarg(z),\\
z &= e^{\ln z} = e^{\ln|z|}e^{iarg(z)} = |z|\frac{z}{|z|} = r(x+iy) = r(\cot\theta + i\csc\theta)\\
&= \ln e^z = \ln|e^z| + iarg(e^z) = \ln e^{rx} + iry.
\end{aligned}$$

3. Half-angle formulas, the metric and time

Half-angle formulas can pertain to the metric [6, pp. 312–313] [10] [11]

$$\tan\frac{\psi}{2} = \cot\left(\frac{\pi}{2}-\frac{\psi}{2}\right) = \tanh\frac{\ln f}{2} \qquad \tan\frac{\theta}{2} = \tan\left(\frac{\pi}{4}-\frac{\psi}{2}\right) = \cot\left(\frac{\pi}{4}+\frac{\psi}{2}\right) = \frac{1}{f}$$
$$\cot\frac{\psi}{2} = -\tan\left(\frac{\pi}{2}+\frac{\psi}{2}\right) = \coth\frac{\ln f}{2} \qquad \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right) = \cot\left(\frac{\pi}{4}-\frac{\psi}{2}\right) = f.$$

The significance of $\cot\frac{\theta}{2} = f$ is that the stereographic projection of a line drawn with an angle $\frac{\theta}{2}$ from the west pole of a sphere, P, and an angle θ from the origin, maps the point $Q = f = \cot\frac{\theta}{2}$ onto the inversive plane $x = 1$ at $\frac{1}{f} = \tan\frac{\theta}{2}$ [7, pp. 92–94] [21, p. 59]. This preserves measuring great circles and angles with a partial metric. Time is understood as a period when $f = x + \sqrt{x^2+1}$ [2, p. 124] [13]

$$\begin{aligned} T &= \frac{1}{f} \\ &= \tan\frac{1}{2}\theta = \tanh\tanh^{-1}\frac{1}{f} = \tanh\frac{1}{2}\ln\frac{1+1/f}{1-1/f} = \tanh\frac{1}{2}\ln\frac{f+1}{f-1} = \tanh\frac{1}{2}\alpha \\ &= \tanh\frac{1}{2}\ln\frac{e^{\sinh^{-1}x}+1}{e^{\sinh^{-1}x}-1}. \end{aligned}$$

4. A unique geometric relationship

The geometric relationship is unique in this scheme. With $T = \frac{1}{f}$ we know that for the vertical unit hyperbola, $y^2 - x^2 = 1$, at the point

$$e^{\alpha/2} = e^{\frac{1}{2}\ln\frac{f+1}{f-1}} = \left(\frac{f+1}{f-1}\right)^{1/2}, \tag{6}$$

the hypotenuse line from the unit circle to that point is

$$\left(\left(\sinh\frac{1}{2}\ln\frac{f+1}{f-1}\right)^2 + \left(\cosh\frac{1}{2}\ln\frac{f+1}{f-1}\right)^2\right)^{1/2} = \left(\frac{f^2+1}{f^2-1}\right)^{1/2}. \tag{7}$$

We use the ratio of both hypotenuses to multiply the circular coordinates

(8)
$$\begin{aligned} \sin\frac{1}{2}\theta &= \left(\frac{1}{f^2+1}\right)^{1/2} &&= \tanh\sinh^{-1}\frac{1}{f} = \frac{1/f}{e^{\sinh^{-1}1/f}-1/f} &&= \frac{e^{i\theta/2}-\cos(\theta/2)}{i} \\ \cos\frac{1}{2}\theta &= \left(\frac{f^2}{f^2+1}\right)^{1/2} &&= \frac{1}{\cosh\sinh^{-1}1/f} = \frac{1}{e^{\sinh^{-1}1/f}-1/f} &&= e^{i\theta/2} - i\sin(\theta/2) \end{aligned}$$

by (7) to derive the hyperbolic coordinates

(9)
$$\begin{aligned} \sinh\frac{1}{2}\alpha &= \left(\frac{1}{f^2-1}\right)^{1/2} &&= \tan\sin^{-1}\frac{1}{f} = \frac{1/f}{e^{i\sin^{-1}1/f}-i1/f} &&= e^{\alpha/2} - \cosh\frac{\alpha}{2} \\ \cosh\frac{1}{2}\alpha &= \left(\frac{f^2}{f^2-1}\right)^{1/2} &&= \frac{1}{\cos\sin^{-1}1/f} = \frac{1}{e^{i\sin^{-1}1/f}-i1/f} &&= e^{\alpha/2} - \sinh\frac{\alpha}{2} \end{aligned}$$

where $\theta = \sin^{-1}\operatorname{sech}\ln f$, an $\alpha = \ln\frac{f+1}{f-1}$, and $\sinh^{-1}\frac{1}{f} = \ln(\frac{1}{f} + \sqrt{\frac{1}{f^2}+1})$.

5. A previous complex trigonometry

A typical depiction is by Harkin and Harkin, who in the April, 2004 Mathematics Magazine, describe a "Geometry of Generalized Complex Numbers." They draw trigonometric functions this way [12, pp. 121–122]:

> "From the point N on the unit circle in C_p drop the perpendicular NP to the radius OM (Figure [2]). At the point M draw a line tangent to the unit circle. Let Q be the point of intersection of the tangent and the line through ON. The lengths of the segments OP, NP, and QM are defined to be the p-cosine (cosp), p-sine (sinp), and p-tangent (tanp), respectively."

Harkin and Harkin use these geometric definitions to equate $\cos\theta$ and $\cosh\alpha$ with cosp, and to equate $\sin\theta$ and $\sinh\alpha$ with sinp. [1]

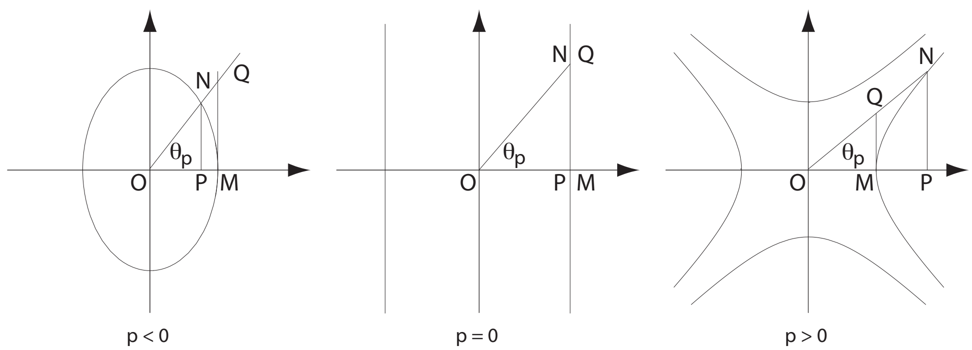

Figure 2. Geometric definitions of cosp, sinp, and tanp

The proportion QM/OM = NP/OP is a common mistake: $\tan 2\frac{\theta}{2} = \frac{2\tan\theta/2}{1-\tan^2\theta/2} = \frac{2\tanh\alpha/2}{1-\tanh^2\alpha/2} = 2\sinh\frac{\alpha}{2}\cosh\frac{\alpha}{2} = \sinh 2\frac{\alpha}{2}$, where $\theta = \sin^{-1}\operatorname{sech}\ln f$ and $\alpha = \ln\frac{f+1}{f-1}$, not $\tan\theta = \tanh\alpha$ [23, p. 67]. Were they to draw the circle together with the vertical hyperbola, as we have done in Figure 1 (bottom), they would have been able to equate circular functions with hyperbolic functions as in equation (3). Specifically, QM $= \tan\theta$ of the unit circle is the same length as $\sinh\alpha$ of the hyperbola, NP $= \sin\theta$ is the same length as $\tanh\alpha$, and OP $= \cos\theta$ is the same length as $\operatorname{sech}\alpha$. For the hyperbola we have OP $= \cosh\alpha = \sec\theta$, and NP = QM $= \sinh\alpha = \tan\theta$. Their drawing hides that the angle $\psi = \angle MOQ \neq \angle PON$ in Figure 1 (top). Half-angle formulas (8) and (9) do maintain the hypotenuse ON as in equation (7), because $\tan\frac{1}{2}\theta = \tanh\frac{1}{2}\alpha$. Another father-and-son team, the Bolyais (1802–60) [4] [10, p. 218] [2, Chapter 3], are famous for finding the geometry of equation (11) in Appendix: the theory of space.

6. A transformation of accelerations for time

This paper's transformation of accelerations is similar yet different from the transformation of velocities of Galileo Galilei (1564–1642)

$$u = u' + v. \tag{10}$$

The quantities are the "absolute velocity" u, *i.e.*, a moving particle's velocity with respect to a fixed reference frame, its "relative velocity" u', *i.e.*, the particle's velocity with respect to a moving reference frame, and the "transport velocity" v, *i.e.*, the velocity of the moving reference frame. The transformation of velocities of Albert Einstein (1879–1955) and H. A. Lorentz (1853–1928) replaced (10) with

$$u = \frac{u' + v}{1 + u'v/c^2}.$$

Einstein believes when $v = c$, c being the speed of light, that $u = c$ rather than $u = c + v$ when the photon $u' = c$ [14, pp. 161, 173, 203–212] [17] [9] [15].

Wave velocity is cyclic, rather than the linear, inertial Newton's first law of motion. This paper's transformation of accelerations is based on harmonic motion. For the derivatives of the time period $T = \tan\frac{1}{2}\theta = \tanh\frac{1}{2}\alpha = \frac{1}{f}$ *second/cycle*

$$\frac{dT}{d\theta} = \frac{dT}{df}\frac{df}{d\theta} = \frac{d}{d\theta}\tan\frac{1}{2}\theta = \frac{d}{d\theta}\tan u = (\sec^2 u)\frac{du}{d\theta} = \frac{1}{2}\sec^2\frac{1}{2}\theta, \quad \theta = \sin^{-1}\operatorname{sech}\ln f$$

$$\frac{dT}{d\alpha} = \frac{dT}{df}\frac{df}{d\alpha} = \frac{d}{d\alpha}\tanh\frac{1}{2}\alpha = \frac{d}{d\alpha}\tanh u = (\operatorname{sech}^2 u)\frac{du}{d\alpha} = \frac{1}{2}\operatorname{sech}^2\frac{1}{2}\alpha, \quad \alpha = \ln\frac{f+1}{f-1}$$

$$\frac{dT}{df} = \frac{dT}{d\alpha}\frac{d\alpha}{df} = \frac{d}{df}f^{-1} = -\frac{1}{f^2}$$

$$\frac{d\alpha}{df} = \frac{d\alpha}{dT}\frac{dT}{df} = \frac{d}{df}\ln\frac{f+1}{f-1} = \frac{1}{u}\frac{du}{df} = (\frac{f-1}{f+1})(\frac{d}{df}\frac{f+1}{f-1}) = (\frac{f-1}{f+1})(\frac{-2}{(f-1)^2}) = -\frac{2}{f^2-1}$$

$$\frac{d\theta}{df} = \frac{d\theta}{dT}\frac{dT}{df} = \frac{d}{df}\sin^{-1}\operatorname{sech}\ln f = \frac{d}{df}\sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}$$

$$= \frac{-((\operatorname{sech}\ln f)(\tanh\ln f))\frac{d}{df}\ln f}{\sqrt{1-(\operatorname{sech}\ln f)^2}} = \frac{-((\operatorname{sech}\ln f)(\tanh\ln f))1/f}{\sqrt{1-(\operatorname{sech}\ln f)^2}}$$

make this paper's transformation of accelerations (refer to Tables 2 and 3),

$$\frac{dT}{d\theta} = \frac{dT}{d\alpha} - \frac{dT}{df} = \frac{dT}{d\alpha} + \frac{1}{f^2}$$

$$u = u' + v$$

$$\text{acceleration}_{\text{absolute}} = \text{acceleration}_{\text{relative}} + \text{acceleration}_{\text{transport}}.$$

If the wave velocity in meters, c, is a constant, then $dc = 0$. But since the wave velocity in radians, θ, is a variable, then we might have $d\theta \neq 0$. This is why we hereby replace c with $\theta = \sin^{-1}\operatorname{sech}\ln f$. The transport acceleration v with $1/f^2$ *second/cycle*^{2}, is also known as the acceleration$_{\text{frame}}$ of the Einsteinian gravitational frame force $F_{\text{frame}} = -\text{mass}\times\text{acceleration}_{\text{frame}}$, in an accelerating, noninertial, rotating and cyclic frame of reference

$$F_\theta = F_\alpha - F_{\text{frame}}$$

$$m\frac{dT}{d\theta} = m\frac{dT}{d\alpha} + m\frac{1}{rf^2}.$$

In an inertial frame of reference there is no frame force.

Wave velocity is like a photon never-ending sine wave with a harmonic motion period, within an unique geometric relationship limited at $e^{\alpha/2}$ (6). When a wave accelerates at $u' = dT/d\alpha = 1/2$ *second/cycle*^{2} relative to a vehicle (reference frame) transporting the wave at no ($v = 0$) acceleration, the wave goes an acceleration relative to a fixed observer at $u = dT/d\theta = 1/2$ *second/cycle*^{2}, within a period $T = 0$ *second/cycle*. As the transporting vehicle accelerates from 0 to the limit $v = 1/f^2 = 1$ *second/cycle*^{2}, the u' acceleration of the wave relative to the vehicle approaches 0, and the fixed observer sees the wave's acceleration at $u = 1$ *second/cycle*^{2}, within a period $T = 1$ *second/cycle*. The transformation of accelerations expresses harmonic motion in special relativity [21].

7. From the metric to the angle of parallelism

Both Lobacevskii and Bolyai discovered the *angle of parallelism*. Rather than discuss the Poincaré disk, this paper describes hyperbolic geometry with hyperbolas. The measurement of great circles and angles with a partial metric preserves the *distance scale* s and the *physical angle of parallelism*, $\Pi(\frac{a}{s}) = 2\tan^{-1} e^{-a/s}$ [2, Chapter 3] [7, pp. 267–268] [8]. We use equation (2) to solve the arc length of a unit circle from $(0, 1)$ to $(\cos\psi, \sin\psi)$, that is $(\operatorname{sech} a, \tanh a)$, when $a = \ln f = \ln(y + \sqrt{y^2 + 1}) = \sinh^{-1} y$ so that

$$\begin{aligned}
(11)\qquad \int_0^{\operatorname{sech} a} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx &= \int_0^a \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt \\
&= \int_0^a ((-\operatorname{sech} t \tanh t)^2 + (\operatorname{sech}^2 t)^2)^{1/2} dt = \int_0^a \frac{2}{e^t + e^{-t}} dt \\
&= \int_1^{e^a} \frac{2}{u + \frac{1}{u}} \frac{du}{u} = 2\tan^{-1} e^a - \frac{\pi}{2} = \psi.
\end{aligned}$$

Solving the s arclength of a horizontal unit hyperbola $x^2 - y^2 = 1$ from $(1, 0)$ to $(\cosh a, \sinh a)$ becomes the distance scale [19, pp. 318–319, 422–423]

$$\begin{aligned}
s = \int_1^{\cosh a} \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx &= \int_0^a \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt \\
&= \int_0^a ((\sinh t)^2 + (\cosh t)^2)^{1/2} dt.
\end{aligned}$$

For the vertical hyperbola, with $a = \ln f = \sinh^{-1} x$, $f = x + \sqrt{x^2 + 1} = e^{\sinh^{-1} x}$, and x increasing from 0 to ∞, we have the angle of parallelism $\Pi(a) = 2\tan^{-1} e^{-a} = 2\tan^{-1} e^{-\sinh^{-1} x} = \theta = \frac{\pi}{2} - \psi = \sin^{-1} \operatorname{sech} \ln f$ equation (3) approaching from $\frac{\pi}{2}$ to 0 radians. Refer to Table 3. With its distance scale s and $a = \sinh^{-1} x$, the physical angle of parallelism $\Pi(\frac{a}{s}) = 2\tan^{-1} e^{-a/s}$ approaches $\frac{\pi}{2}$ radians, implying parallelism [19, pp. 300, 414, 434] [10, p. 217] [7, pp. 315, 377]. The length of $a/s = 1$ has $2\tan^{-1} e^{-1} = 0.70502....$ Refer to Table 2. When $a = \sinh^{-1} 1$ in the distance scale s, the *standard length* has $a = s \times \sinh^{-1} 1$ for $2\tan^{-1} e^{-a/s} = \frac{\pi}{4} = 0.78539....$ The base $a = \ln f = \sinh^{-1} x$ of the astronomical asymptotic triangle CBA is an important new kind of curve which is orthogonal to all parallels BA.

8. The theory of space

As George Martin (1932–?) says [19, pp. 301–302],

> "The largest physical triangles that can be accurately measured are astronomical. Let E stand for the Earth, S for the Sun, and V for the brilliant blue star Vega. $\measuredangle SEV$ can be measured from the Earth when $\measuredangle ESV$ is right. Using this measurement and the fact that [the defect of the triangle] SEV is less than $\frac{\pi}{2}$ - $\measuredangle SEV$, one obtains [the defect of the triangle] $SEV < 0.0000004$. "

Were the physical angle of parallelism for Vega determined by $e^{7.8}$ with 7.8 parsecs (pc), the angle would agree more closely with Martin's evidence by $\Pi(\frac{a}{s}) = \frac{\pi}{2} - 0.0000000291 = 1.570796298$ at $a = \ln f = 20.73$ geometrical units. Poincaré uses "the defect of the triangle" in defining the angle of parallelism, while Martin's innovation is the *distance scale s*.

Compare this to one *parsec* (pc) to define the distance of the Earth to the Sun (1 *astronomical unit*, or AU) that subtends an angle of 1 second of arc, equivalent to 206265 astronomical units [5, pp. 38, 380, 526, A-5][22, p. 347]. The Parallactic angle, or *parallax*, conventionally defined to be half the star's apparent shift relative to the background stars as we move from one side of the Earth's orbit to the other, measures $\measuredangle SAE$, the distance to (any) star in parsecs. The parallax (in arc seconds) decreases as distance (in parsecs) increases. If the geometric electromagnetic spectrum, Table 1 (below), holds, 206265 AU is like a frequency $f = x + \sqrt{x^2+1}$ cycles/second with an $x = e^{pc} \times 206264.8062 = e^{pc} \times 648000/\pi$, a wavelength $r = \theta/f$ radian/cycle, and a wave velocity $\theta = \sin^{-1} \operatorname{sech} \ln f$ radian/second.

The base $a = 1$ unit of the Sun to the Earth has 206264.8 units from the Sun to the idealized star A with $\measuredangle SAE = 1$ arc second. For the Sun to the star Vega, the $\measuredangle SVE = 1/7.8$ arc seconds has a distance $SV = 7.8 \times 206264.8$ units of 7.8 parsecs.

With hyperbolic geometry we know the physical angle of parallelism $\Pi(\frac{a}{s})$ for understanding the asymptotic triangle. An asymptotic triangle with a physical angle of parallelism $\measuredangle CBA = \Pi(\frac{a}{s})$ has a base $a = \ln f = CB$ *geometrical units* from point C to point B, a $\measuredangle CAB = 0$, and a distance $BA = \infty$. For Vega the $\measuredangle SEV = \Pi(\frac{a}{s}) = \frac{\pi}{2} - (2.91 \times 10^{-8})$ radians has a base $SE = a = 20.73$ geometrical units, a $\measuredangle SVE = 0$, and an $EV = \infty$. The author hypothesizes $\Pi(\frac{a}{s})$ connects to 7.8 parsecs by an exponential of 7.8. Refer to Table 2.

To traverse from the Sun to the center of our galaxy takes 8000 parsecs. The galaxy is roughly 30000 parsecs wide. It may be impractical to use the angle of parallelism, with its exponentiation, to describe great distances. But when we realize $x = e^{pc} \times 648000/\pi$, therefore $f = x + \sqrt{x^2+1}$, we can construct an electromagnetic spectrum which geometrically measures the near universe [23, pp. 23–37, 208, 305–311].

An asymptotic triangle with a base $a = CB = \ln f = \ln(x + \sqrt{x^2+1})$ is listed in Table 3. The unique geometric relationship describes how $e^{\alpha/2}$ limits values on the hyperbola. The asymptotic triangle illustrates space. The transformation of accelerations illustrates time.

9. A geometric electromagnetic spectrum

The following Tables 1 and 2 have eleven categories, formulated

$$
\begin{aligned}
\text{wavevelocymeters} &= c = 299792458\frac{\text{meters}}{\text{second}} = \lambda\frac{\text{meters}}{\text{cycle}}\times f\frac{\text{cycles}}{\text{second}}\\
\text{wavevelocyradian} &= \theta\frac{\text{radian}}{\text{second}} = r\frac{\text{radian}}{\text{cycle}}\times f\frac{\text{cycles}}{\text{second}} = \sin^{-1}\operatorname{sech}\ln f\\
x &= e^{parsecs}\times 648000/\pi\\
\text{frequency} &= f\frac{\text{cycles}}{\text{second}} = x+\sqrt{x^2+1}\\
\text{wavelengthradian} &= r\frac{\text{radian}}{\text{cycle}} = \theta\frac{\text{radian}}{\text{second}}\times\frac{1}{f}\frac{\text{second}}{\text{cycle}}\\
\text{wavelengthmeters} &= \lambda\frac{\text{meters}}{\text{cycle}} = 299792458\frac{\text{meters}}{\text{second}}\times\frac{1}{f}\frac{\text{second}}{\text{cycle}}\\
\Pi(\frac{a}{s}) &= 2\tan^{-1}e^{-a/s},\quad a=\ln f,\quad s=\int_0^a((\sinh t)^2+(\cosh t)^2)^{1/2}dt\\
\text{period} &= T = \frac{1}{f}\frac{\text{second}}{\text{cycle}} = \tan\frac{1}{2}\theta = \tanh\frac{1}{2}\alpha\\
\text{absoluteaccel} &= u = \frac{dT}{d\theta} = \frac{1}{2}\sec^2\frac{1}{2}\theta\quad\frac{\text{second}}{\text{cycle}^2},\quad \theta=\sin^{-1}\operatorname{sech}\ln f\\
\text{relativeaccel} &= u' = \frac{dT}{d\alpha} = \frac{1}{2}\operatorname{sech}^2\frac{1}{2}\alpha\quad\frac{\text{second}}{\text{cycle}^2},\quad \alpha=\ln\frac{f+1}{f-1}\\
\text{transportaccel} &= v = 1/f^2\quad\frac{\text{second}}{\text{cycle}^2}.
\end{aligned}
$$

Louis de Broglie's (1892–1987) wavelength $\lambda = h/p$ has h, Planck's constant, and $p = E/c$, a photon's momentum p with energy E at the velocity of light c. Einstein relates frequency f cycles/second $= 1/\lambda$ cycle/meter $\times$ c meters/second to photon energy $E = cp$ by $E = hf = hc/\lambda$. Thus we have $\lambda = c/f$ meters/cycle $= c/(E/h) = hc/E = hc/cp = h/p$. Using $f = \theta/r$ with our wave velocity $\theta =$ radian/second measure of c, we have $E = \theta p = hf = h\theta/r$. This solves to $r = \theta/f$ radian/cycle $= \theta/(E/h) = h\theta/E = h\theta/\theta p = h/p$.

10. Conclusion

By Ockham's Razor, the simpler theorem is preferred. Our hyperbolic trigonometric functions define circular trigonometric functions, providing a hyperbolic alternative to Euler's formula (4). Double-angle formulas appear in complex products and roots. Frequency defines the metric period. Rather than with the Newtonian linear velocity, force accelerations are understood with derivatives of a harmonic motion cyclic periodic wave velocity. The angle of parallelism is made with an important curve a. The physical angle of parallelism with a and distance scale s is hypothesized to be determined by the parallactic exponential of parsecs. We may use this geometry to describe the electromagnetic spectrum. The Appendix adds the Lorentz transformation and an algebra of distance and time. In these pages we have sought to measure astronomical quantities with hyperbolic geometry geodesics in special relativity and quantum mechanics.

References

1. L. V. Ahlfors, *Complex analysis*, 3rd ed., McGraw-Hill, New York, 1979.
2. J. W. Anderson, *Hyperbolic geometry*, Springer-Verlag, London, 2005.
3. W. H. Beyer, *Crc handbook of mathematical sciences*, CRC Press, Boca Raton, 1987.
4. J. Bolyai, *Appendix: the theory of space*, no. 138, North-Holland Mathematical Studies, Amsterdam, 1987.
5. E. Chaisson, *Astronomy today*, Prentice Hall, London, 1999.
6. G. Chrystal, *Algebra*, vol. 2, Black, London, 1931.
7. H. S. M. Coxeter, *Introduction to geometry*, 2nd ed., Univ. Toronto Press, Toronto, 1961.
8. ———, *Parallel lines*, Canad. Math. Bull. **21** (1978), 385–397.
9. A. Einstein, *Relativity, the special and the general theory*, 15th ed., Methuen, London, 1957.
10. R. C. Eskew, *Time, gravity and the exterior angle of parallelism*, Speculations in Science and Technology **21** (1999), 213–225, www.wkap.nl.
11. ———, *Standardizing distance and time*, Los Alamos e-prints (2001), no. abs/physics/0107005, 1–10, arxiv.org.
12. A. A. Harkin and J. B. Harkin, *Geometry of generalized complex numbers*, Math. Mag. **77** (2004), no. 2, 118–129.
13. S. W. Hawking and R. Penrose, *The nature of space and time*, Princeton University Press, Princeton, NJ, 1996.
14. I. M. Iaglom, *A simple non-euclidean geometry and its physical basis*, Nauka, Moscow, 1969 (Russian), English translation available.
15. J. D. Jackson, *Classical electrodynamics*, 3rd ed., Wiley, New York, 1999.
16. N. I. Lobachevskii, *Geometrical researches on the theory of parallels*, Karzan, Berlin, 1840 (Russian), English translation available.
17. H. A. Lorentz, *The principle of relativity*, Dover, New York, 1923.
18. J. E. Marsden, *Basic complex analysis*, W. H. Freeman and Company, New York, 1987.
19. G. E. Martin, *The foundations of geometry and the non-euclidean plane*, Intext Educational, New York, 1972.
20. B. P. Palka, *An introduction to complex function theory*, Springer-Verlag, New York, 1991.
21. W. Rindler, *Essential relativity: special, general, and cosmological*, 2nd ed., Van Nostrand Reinhold, New York, 1969.
22. D. A. Teets, *Transits of venus and the astronomical unit*, Math. Mag. **76** (2003), no. 5, 335–348.
23. C. W. Misner K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1970.

Table 1. Geometric electromagnetic spectrum in meters and radians.

x	frequency f *cycles/second*	wavelength r *radian/cycle*	wavelengt λ *meter/cycle*	wave velocity θ *radian/second*	Name
0	1.0	$\pi/2 = 1.5707$	299792458	$\pi/2 = 1.57079$	
1.0	$1+\sqrt{2} = 2.41$	0.325322571	124178102	$\pi/4 = 0.78539$	
$e^0 \times 648000/\pi$	412529.6125	1.17×10^{-11}	726.7174256	0.000004848	heat
$e^1 \times 648000/\pi$	1121371.749	1.59×10^{-12}	267.3444005	0.000001784	AM
$e^3 \times 648000/\pi$	8285878.764	2.91×10^{-14}	36.18113015	0.000000241	
$e^5 \times 648000/\pi$	61224823.01	5.33×10^{-16}	4.896583498	0.000000033	
$e^7 \times 648000/\pi$	452393651.9	9.77×10^{-18}	0.662680515	0.000000004	FM
$e^9 \times 648000/\pi$	3342762073	1.78×10^{-19}	0.089684055	5.98×10^{-10}	
$e^{11} \times 648000/\pi$	2.46×10^{10}	3.27×10^{-21}	0.012137417	8.09×10^{-11}	radar
$e^{13} \times 648000/\pi$	1.82×10^{11}	6.00×10^{-23}	0.001642621	1.09×10^{-11}	
$e^{15} \times 648000/\pi$	1.34×10^{12}	1.09×10^{-24}	0.000222305	1.48×10^{-12}	Mic W
$e^{17} \times 648000/\pi$	9.96×10^{12}	2.01×10^{-26}	0.000030086	2.00×10^{-13}	
$e^{19} \times 648000/\pi$	7.36×10^{13}	3.68×10^{-28}	0.000004072	2.71×10^{-14}	Infred
$e^{21} \times 648000/\pi$	5.44×10^{14}	6.75×10^{-30}	0.000000551	3.67×10^{-15}	Visible
$e^{23} \times 648000/\pi$	4.02×10^{15}	1.23×10^{-31}	0.000000075	4.97×10^{-16}	UV
$e^{25} \times 648000/\pi$	2.97×10^{16}	2.26×10^{-33}	0.000000001	6.73×10^{-17}	
$e^{27} \times 648000/\pi$	2.19×10^{17}	4.15×10^{-35}	0.000000001	9.11×10^{-18}	X-ray
$e^{29} \times 648000/\pi$	1.62×10^{18}	7.60×10^{-37}	1.84×10^{-10}	1.23×10^{-18}	atom
$e^{31} \times 648000/\pi$	1.19×10^{19}	1.39×10^{-38}	2.50×10^{-11}	1.66×10^{-19}	
$e^{33} \times 648000/\pi$	8.85×10^{19}	2.55×10^{-40}	3.38×10^{-12}	2.25×10^{-20}	γ - ray
$e^{35} \times 648000/\pi$	6.54×10^{20}	4.67×10^{-42}	4.58×10^{-13}	3.05×10^{-21}	
$e^{37} \times 648000/\pi$	4.83×10^{21}	8.55×10^{-44}	6.20×10^{-14}	4.13×10^{-22}	
$e^{39} \times 648000/\pi$	1.78×10^{22}	2.31×10^{-44}	8.39×10^{-15}	3.57×10^{-22}	photon
$e^{41} \times 648000/\pi$	2.63×10^{23}	2.87×10^{-47}	1.13×10^{-15}	7.57×10^{-24}	
$e^{43} \times 648000/\pi$	1.95×10^{24}	5.25×10^{-49}	1.53×10^{-16}	1.02×10^{-24}	nucleus
$e^{45} \times 648000/\pi$	1.44×10^{25}	9.62×10^{-51}	2.08×10^{-17}	1.38×10^{-25}	
$e^{47} \times 648000/\pi$	1.06×10^{26}	1.76×10^{-52}	2.81×10^{-18}	1.87×10^{-26}	strings
$e^{49} \times 648000/\pi$	7.86×10^{26}	3.23×10^{-54}	3.81×10^{-19}	2.54×10^{-27}	
$e^{51} \times 648000/\pi$	5.81×10^{27}	5.91×10^{-56}	5.15×10^{-20}	3.43×10^{-28}	
$e^{53} \times 648000/\pi$	4.29×10^{28}	1.08×10^{-57}	6.97×10^{-21}	4.65×10^{-29}	
$e^{55} \times 648000/\pi$	3.17×10^{29}	1.98×10^{-59}	9.44×10^{-22}	6.30×10^{-30}	
$e^{57} \times 648000/\pi$	2.34×10^{30}	3.63×10^{-61}	1.27×10^{-22}	8.52×10^{-31}	
$e^{86.7} \times 206265$	1.87×10^{43}	5.69×10^{-87}	1.61×10^{-35}	1.06×10^{-43}	Planck
∞	∞	0	0	0	

Table 2. Physical angle of parallelism and addition of accelerations as periods.

phy angle of parallsm $\Pi(\frac{a}{s}) = 2\tan^{-1} e^{-a/s}$	time $T = 1/f$ *second/cycle*	absolute acceleration $u = dT/d\theta$	relative acceleration $u' = dT/d\alpha$	transport accl $v = 1/f^2$
0.705026844	1.0	1.0	0	1.0
0.843488088	$\sqrt{2} - 1 = 0.41$	$0.5 + 0.085786$	$0.5 - 0.085786$	0.171572875
1.570752000	0.000002424	$0.5 + (2.93 \times 10^{-12})$	$0.5 - (2.93 \times 10^{-12})$	5.87×10^{-12}
1.570778759	0.000000892	$0.5 + (3.97 \times 10^{-13})$	$0.5 - (3.97 \times 10^{-13})$	7.95×10^{-13}
1.570793608	0.000000121	$0.5 + (7.32 \times 10^{-15})$	$0.5 - (7.32 \times 10^{-15})$	1.45×10^{-14}
1.570795913	0.000000016	$0.5 + (2.22 \times 10^{-16})$	$0.5 - (2.22 \times 10^{-16})$	2.66×10^{-16}
$\pi/2 - (6.23 \times 10^{-8})$	0.000000002	$0.5 + d$	$0.5 - d$	4.88×10^{-18}
$\pi/2 - (9.27 \times 10^{-9})$	2.99×10^{-10}	$0.5 + d$	$0.5 - d$	8.94×10^{-20}
$\pi/2 - (1.37 \times 10^{-9})$	4.04×10^{-11}	$0.5 + d$	$0.5 - d$	1.63×10^{-21}
$\pi/2 - (2.00 \times 10^{-10})$	5.47×10^{-12}	$0.5 + d$	$0.5 - d$	3.00×10^{-23}
$\pi/2 - (2.92 \times 10^{-11})$	7.41×10^{-13}	$0.5 + d$	$0.5 - d$	5.49×10^{-25}
$\pi/2 - (4.24 \times 10^{-12})$	1.00×10^{-13}	$0.5 + d$	$0.5 - d$	1.00×10^{-26}
$\pi/2 - (6.13 \times 10^{-13})$	1.35×10^{-14}	$0.5 + d$	$0.5 - d$	1.84×10^{-28}
$\pi/2 - (8.81 \times 10^{-14})$	1.83×10^{-15}	$0.5 + d$	$0.5 - d$	3.37×10^{-30}
$\pi/2 - (1.26 \times 10^{-14})$	2.48×10^{-16}	$0.5 + d$	$0.5 - d$	6.18×10^{-32}
$\pi/2 - (1.80 \times 10^{-15})$	3.36×10^{-17}	$0.5 + d$	$0.5 - d$	1.13×10^{-33}
$\pi/2 - (2.57 \times 10^{-16})$	4.55×10^{-18}	$0.5 + d$	$0.5 - d$	2.07×10^{-35}
$\pi/2 - (3.65 \times 10^{-17})$	6.16×10^{-19}	$0.5 + d$	$0.5 - d$	3.80×10^{-37}
$\pi/2 - (5.18 \times 10^{-18})$	8.34×10^{-20}	$0.5 + d$	$0.5 - d$	6.96×10^{-39}
$\pi/2 - (7.33 \times 10^{-19})$	1.12×10^{-20}	$0.5 + d$	$0.5 - d$	1.27×10^{-40}
$\pi/2 - (1.03 \times 10^{-19})$	1.52×10^{-21}	$0.5 + d$	$0.5 - d$	2.33×10^{-42}
$\pi/2 - (1.46 \times 10^{-20})$	2.06×10^{-22}	$0.5 + d$	$0.5 - d$	4.27×10^{-44}
$\pi/2 - (2.05 \times 10^{-21})$	2.79×10^{-23}	$0.5 + d$	$0.5 - d$	7.83×10^{-46}
$\pi/2 - (2.88 \times 10^{-22})$	3.78×10^{-24}	$0.5 + d$	$0.5 - d$	2.62×10^{-47}
$\pi/2 - (4.05 \times 10^{-23})$	5.12×10^{-25}	$0.5 + d$	$0.5 - d$	6.96×10^{-49}
$\pi/2 - (5.68 \times 10^{-24})$	6.93×10^{-26}	$0.5 + d$	$0.5 - d$	4.81×10^{-51}
$\pi/2 - (7.95 \times 10^{-25})$	9.39×10^{-27}	$0.5 + d$	$0.5 - d$	8.81×10^{-53}
$\pi/2 - (1.11 \times 10^{-25})$	1.27×10^{-27}	$0.5 + d$	$0.5 - d$	1.61×10^{-54}
$\pi/2 - (1.55 \times 10^{-26})$	1.71×10^{-28}	$0.5 + d$	$0.5 - d$	2.95×10^{-56}
$\pi/2 - (2.17 \times 10^{-27})$	2.32×10^{-29}	$0.5 + d$	$0.5 - d$	5.41×10^{-58}
$\pi/2 - (3.02 \times 10^{-28})$	3.15×10^{-30}	$0.5 + d$	$0.5 - d$	9.92×10^{-60}
$\pi/2 - (4.21 \times 10^{-29})$	4.26×10^{-31}	$0.5 + d$	$0.5 - d$	1.81×10^{-61}
$-$	5.38×10^{-44}	$0.5 + d$	$0.5 - d$	2.90×10^{-87}
$\pi/2 = 1.570796327$	0	0.5	0.5	0

Table 3. $\Pi(a) = 2\tan^{-1} e^{-a} = \theta$ values

x	$CB = a = \ln f$	$u = dT/d\theta$	$u' = dT/d\alpha$	$v = 1/f^2$	$\Pi(a)$
0	$\ln(0 + \sqrt{0^2 + 1})$	1.0	0.0	1.0	$\pi/2 = 1.5707$
1	$\ln(1 + \sqrt{1^2 + 1})$	$0.5 + .085786438$	$0.5 - .085786438$	0.17157287	$\pi/4 = 0.7853$
2	$\ln(2 + \sqrt{2^2 + 1})$	$0.5 + .027864045$	$0.5 - .027864045$	0.05572809	0.463647609
3	$\ln(3 + \sqrt{3^2 + 1})$	$0.5 + .013167019$	$0.5 - .013167019$	0.02633403	0.321750554
n	$\ln(n + \sqrt{n^2 + 1})$	0.5	0.5	0	0

Appendix . Changing the Lorentz transformation

With the concept of *proper time*, a moving wave with instantaneous velocity (*i.e.*, with "transport acceleration" of a moving reference frame) $v(t) = \frac{1}{f^2}(t)$ relative to some inertial system K (*i.e.*, with "absolute acceleration" $dT/d\theta$ with respect to a fixed reference frame) changes its position in a time interval dt by $dx = vdt = \frac{1}{f^2}dt$. The space and time coordinates in K', $(t', z', x', y') = (x_0^{'}, x_1^{'}, x_2^{'}, x_3^{'}) = (ct', z', x', y')$, where the system is instantaneously at rest (*i.e.*, with the wave's "relative acceleration" $dT/d\alpha$ with respect to the moving reference frame), are related to those in K, $(t, z, x, y) = (x_0, x_1, x_2, x_3) = (ct, z, x, y)$, by the inverse *Lorentz transformation*

$$
\begin{aligned}
x_0 &= \gamma(x_0^{'} + x_1^{'}\beta) = (\cosh a)(x_0^{'} + x_1^{'} \tanh a) \\
x_1 &= \gamma(x_1^{'} + x_0^{'}\beta) = (\cosh a)(x_1^{'} + x_0^{'} \tanh a) \\
x_2 &= x_2^{'} \\
x_3 &= x_3^{'}.
\end{aligned}
$$

With the *boost parameter* ξ it said that

$$
\begin{aligned}
\beta &= \tanh \xi \\
\gamma &= \cosh \xi \\
\gamma\beta &= \sinh \xi
\end{aligned}
$$

applies to

$$
\begin{aligned}
x_0^{'} &= x_0 \cosh \xi - x_1 \sinh \xi \\
x_1^{'} &= -x_0 \sinh \xi + x_1 \cosh \xi.
\end{aligned}
\tag{12}
$$

It is Einstein's thought that $c = 1$ with the velocity v in $\tanh \xi = \frac{v}{c}$. However, the wave velocity θ, period $T = \frac{1}{f}$, and transport acceleration $v = 1/f^2$ relate differing wave velocities with a differing fractions of a second t. Rather than using the boost parameter ξ in (12), the hyperbolic coordinates are equated with the circular coordinates by $T = 1/f = \tan \frac{1}{2}\theta = \tanh \frac{1}{2}\alpha$. The Lorentz time t equalling the distance over θ of the $x_0^{'}$ observer becomes

$$
\begin{aligned}
x_0^{'} &= x_0 \cosh \frac{1}{2} \ln \frac{f+1}{f-1} - x_1 \sinh \frac{1}{2} \ln \frac{f+1}{f-1} \\
&= \left(\cosh \frac{1}{2} \ln \frac{f+1}{f-1}\right) \left(x_0 - x_1 \tanh \frac{1}{2} \ln \frac{f+1}{f-1}\right) \\
&= \left(\frac{f^2}{f^2-1}\right)^{1/2} \left(x_0 - x_1 \frac{1}{f}\right) \\
x_1^{'} &= -x_0 \sinh \frac{1}{2} \ln \frac{f+1}{f-1} + x_1 \cosh \frac{1}{2} \ln \frac{f+1}{f-1} \\
&= \left(\cosh \frac{1}{2} \ln \frac{f+1}{f-1}\right) \left(x_1 - x_0 \tanh \frac{1}{2} \ln \frac{f+1}{f-1}\right) \\
&= \left(\frac{f^2}{f^2-1}\right)^{1/2} \left(x_1 - x_0 \frac{1}{f}\right).
\end{aligned}
$$

The moving wave has advanced a distance $vdt = \frac{1}{f^2}dt = dx_0^{'}$. Time is measured with twice the distance L of the hypotenuse, vs. twice the height D of the side of the triangle. The *proper time* observer sees $x_0 = 2D/(\frac{1}{f})$. The $x_0^{'}$ observer, however, sees $x_0^{'} = 2L/(\frac{1}{f})$, where $L = (((\frac{1}{f^2}x_0^{'})/2)^2 + D^2)^{1/2}$ and $D = (\frac{1}{f}x_0)/2$, by which

$$x_0^{'} = \left(\frac{f^2}{f^2-1}\right)^{1/2} x_0$$

is derivable when $x_1 = 0$ occurs simultaneously. Moving clocks run slow. Events will be separated by the time interval

$$x_0^{'} = \left(\frac{f^2}{f^2-1}\right)^{1/2} (-x_1\frac{1}{f})$$

since $x_0 = 0$, although the events are simultaneous in time [21, p. 32].

The element of arc length ds of the wave's path has $ds^2 = c^2dt^2 - |dx|^2$ when $dx_1 = r\sin\psi ds$, $dx_2 = \cos\psi ds$, $dx_3 = \sin\psi ds$ are "increments" of x_1, x_2, x_3 having the angle ψ. The direction of the path curve's polar-equation-tangent determined by the angle ϕ which this tangent makes with the radius r or by the angle $\psi = \theta + \phi$ which it makes with the x-axis thereby constructs $dr = \cos\phi ds$, $rd\varphi = \sin\phi ds$, and $r\sin\varphi d\theta$ [7, pp. 120–121]. A motionless wave has $ds^2 = dt^2$. The new distance s is called *proper time* τ, and the *Lorentz metric* is $d\tau^2 = c^2dt^2 - r^2\sin^2\varphi d\theta^2 - dr^2 - r^2d\varphi^2$.

The square of the corresponding infinitesimal invariant interval ds is

$$\begin{aligned} ds^2 &= c^2dt^2 - |dx|^2 \\ &= c^2dt^2(1-\beta^2) \\ &= \frac{1}{f^2}dt^2\left(\frac{f^2-1}{f^2}\right) \end{aligned}$$

where $\beta = \frac{v}{c} = \tanh a$ or where c is replaced by $\frac{1}{f} = \tanh\frac{1}{2}\ln\frac{f+1}{f-1}$ and v by $\frac{1}{f^2}$. In the coordinate system K' where the system is instantaneously at rest, the space-time increments are $dt' = d\tau$, $dx' = 0$. Thus the invariant interval is $ds = cd\tau$ or $ds = \frac{1}{f}d\tau$. The increment of time $d\tau$ in the instantaneous rest frame of the system is an invariant quantity that takes the form

$$d\tau = dt(1-\beta^2(t))^{1/2} = \frac{dt}{\gamma(t)}$$

$$d\tau = dt\left(\frac{f^2}{f^2-1}(t)\right)^{-1/2}$$

where $\gamma = \cosh a = (1-\beta^2)^{-1/2}$ and $\cosh\frac{1}{2}\ln\frac{f+1}{f-1} = (\frac{f^2}{f^2-1})^{1/2}$. That is the time as seen in the rest frame of the system [15, pp. 524–528].

Appendix . The algebra of distance and time

The following algebraic equations illustrate the theory of distance and time:

$$A = \frac{F}{D} = \frac{E}{F} = \frac{G}{CF} = \frac{648000}{\pi} = 206264.8062... \neq 206265$$

is the number of astronomical units in 1 parsec $= \frac{1}{1arcsec}$, the number of seconds in 1 radian, and the number x of $x + \sqrt{x^2+1}$ cycles in 1 second.

$$B = CF = DH = \frac{G}{A} = 149597870660$$

is the number of meters in 1 AU.

$$C = 299792458$$

is the number of meters in 1 second.

$$D = \frac{E}{A^2} = \frac{F}{A} = \frac{G}{A^2C} = \frac{B}{AC} = \frac{10685562190\pi}{13876108056000} = 0.00241924346078189...$$

is the number of radians in 1 AU.

$$E = A^2D = AF = \frac{G}{C} = \frac{6924244299120000}{21413747\pi} = 1.02927125026818... \times 10^8$$

is the number of seconds in $A = 206264.8$ astronomical units, equivalent to $(Esec)$

$(\frac{1}{31471200}\frac{lyear}{sec}) = 3.27$ light-years, and the number of $x + \sqrt{x^2+1}$ cycles in 1 AU.

$$F = AD = \frac{E}{A} = \frac{G}{H} = \frac{10685562190}{21413747} = 499.004783702731$$

is the number of radians in $A = 206264.8$ astronomical units, and the number of seconds in 1 AU.

$$G = AB = CE = FH = \frac{96939420187680000}{\pi} = 3.08567758066630805... \times 10^{16}$$

is the number of meters in $A = 206264.8$ astronomical units.

$$H = AC = \frac{B}{D} = \frac{194265512784000}{\pi} = 6.18366332637108... \times 10^{13}$$

is the number of meters in 1 radian.

Assuming A, B and C derives D, E, F, G and H. The $radian/GU$ length of a *geometrical unit*, GU, when $x = A$, is the number

$$D\frac{radian}{AU} \times \frac{x}{\ln(x+\sqrt{x^2+1})}\frac{AU}{GU} = 38.5925945....$$

Other units are

$$\begin{aligned}
1sec &= \frac{1}{F}astunit \\
&= \frac{1}{A}radian \\
1radian &= \frac{1}{D}astunits \qquad 2\pi radians &&= \frac{2\pi}{D}astunits \\
&= Asec &&= 1296000sec = 15days \\
&= Hmeters &&= 388531025568000meters \\
&= A^2cycles \\
x+\sqrt{x^2+1}cycles &= \frac{1}{A}sec \\
&= \frac{1}{A^2}radian \\
&= \frac{C}{A}meters
\end{aligned}$$

$$\begin{aligned}
(A\frac{cycles}{sec})(F\frac{sec}{astunit}) &= E\frac{cycles}{astunit} \\
A\frac{cycles}{sec} &= (E\frac{cycles}{astunit})(\frac{1}{F}\frac{astunit}{sec}) \\
A\frac{cycles}{sec} &= (\frac{1}{A}\frac{radian}{sec})(A^2\frac{cycles}{radian}) \\
(E\frac{cycles}{astunit})(\frac{1}{F}\frac{astunit}{sec}) &= (\frac{1}{A}\frac{radian}{sec})(A^2\frac{cycles}{radian}) \\
(E\frac{cycles}{astunit})(\frac{1}{A^2}\frac{radian}{cycle})(\frac{1}{F}\frac{astunit}{sec}) &= \frac{1}{A}\frac{radian}{sec} \\
(D\frac{radian}{astunit})(\frac{1}{F}\frac{astunit}{sec}) &= \frac{1}{A}\frac{radian}{sec}.
\end{aligned}$$

The *period* $T = 1/f$ second/cycle has $f \geq 1$. Our algebra creates multiples of $1/A$ second/$(x+\sqrt{x^2+1})$ cycle. The algebra has plural A seconds/radian, F seconds/astronomical unit, and E seconds/Aastronomical units. A coordinate system of 1296000 seconds = 15 days = 14 full days = 388531025568000 meters = 360 degrees × 3600 arcseconds in a circle is probable.

PO Box 8150, Austin, Texas 78713-8150, USA
E-mail address: eskewr@io.com